Discrete-Time LTI Systems

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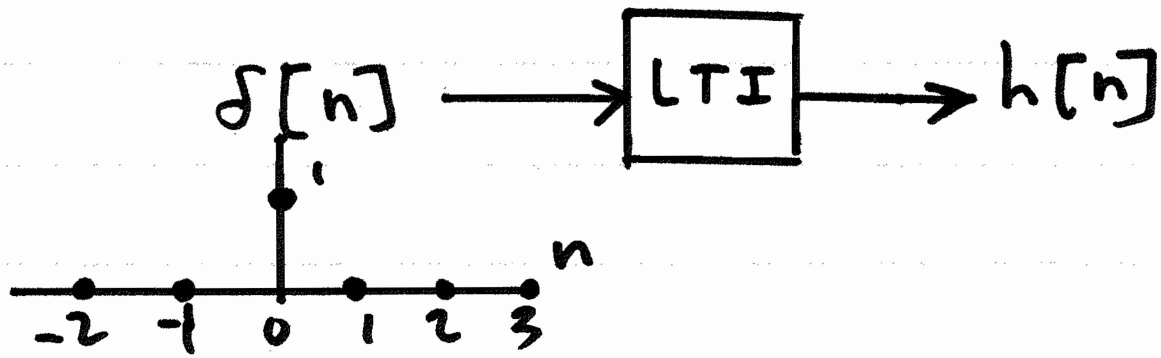
- DT Convolution

- Define impulse response of DT System that is both Linear and Time-Invariant (LTI)



- To easily derive DT convolution formula, we view $x[n]$ (input) as a sum of amplitude-scaled and time-shifted (Kronecker) Delta functions $\Rightarrow$ See Fig. 2.1 on pg. 76

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

- Time-Invariance dictates:

$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

- Homogeneity aspect of linearity dictates:

$$x[k]\,\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k]\,h[n-k]$$

- Superposition aspect of linearity dictates:

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta(n-k) \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= x[n] \qquad\qquad = x[n] * h[n]$$

See Fig. 2.2 on pg. 79

- Summarizing:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

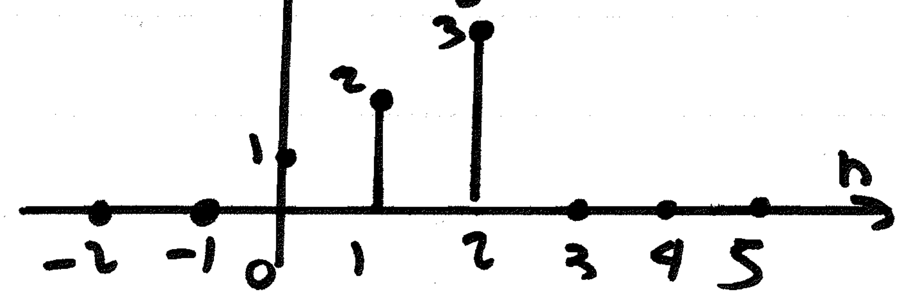
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- There are at least 3 ways to compute DT convolution:
- Method 1: collectively sum

$$y[n] = \ldots + x[-1]\,h[n+1] + x[0]\,h[n] + x[1]\,h[n-1] + x[2]\,h[n-2] + \ldots$$

- Example: $y[n] = x[n] + x[n-1] + x[n-2]$

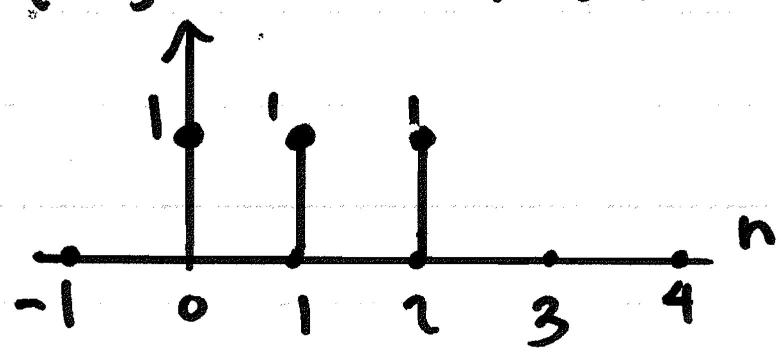
- Find output when: $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$



- Impulse response of system?

$y[n] = h[n]$ when $x[n] = \delta[n]$

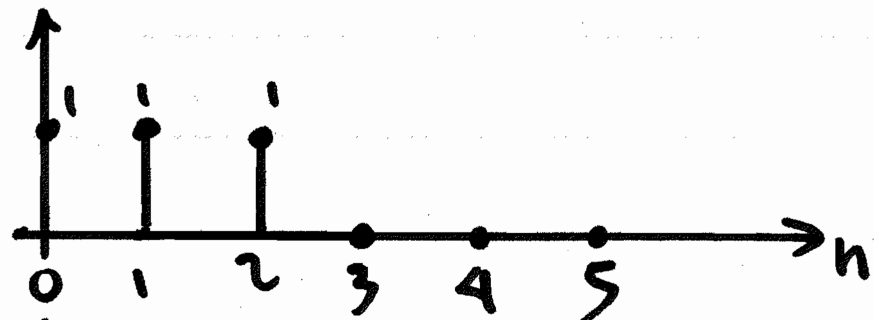
$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$



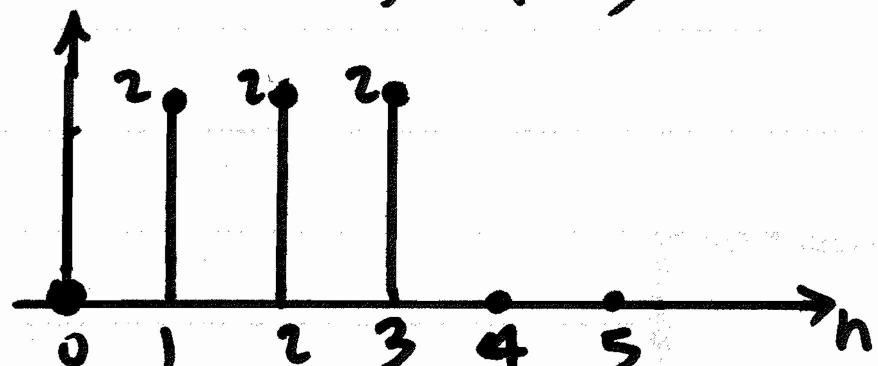
• Since $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

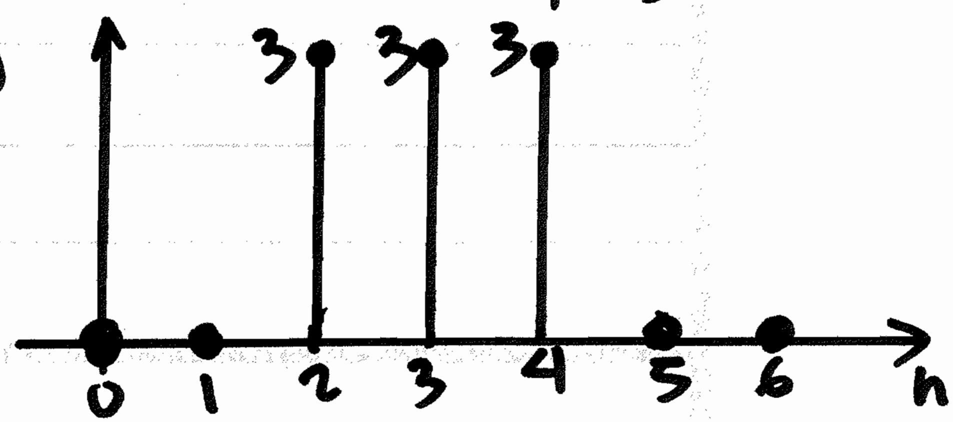
$= x[0]\,h[n] = h[n]$



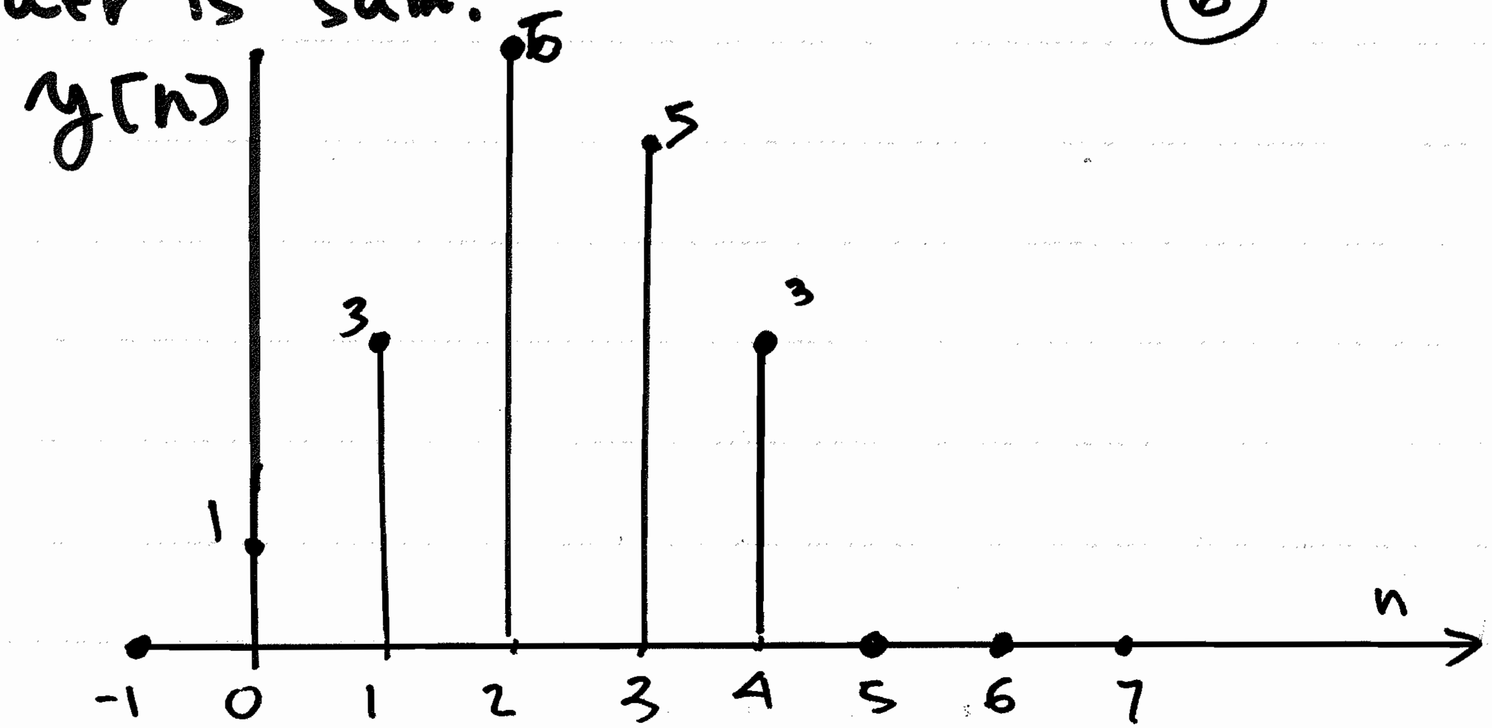
$+\ x[1]\,h[n-1] = 2h[n-1]$



$+\ x[2]\,h[n-2] = 3h[n-2]$



• Answer is sum: ⑥



x[n] of "length" 3
h[n] of "length" 3
} x[n] * h[n] of length 3+3-1=5

Generally: x[n] of "length" N_1
h[n] of "length" N_2
} x[n] * h[n] of "length" $N_1 + N_2 - 1$

Note: not concerned with initial conditions in this course ⇒ unless stated otherwise assume system is initially at rest ⇒ all initial conditions = 0

Method 2: "run" input signal thru difference equation (**Note:** all DT LTI systems may be expressed as a difference equation)

- In the previous example: $x[n] = 0$ for $n < 0$

$x[0] = 1$, $x[1] = 2$, $x[2] = 3$, $x[n] = 0$ for $n > 2$

$n = 0$ $\quad y[0] = x[0] + x[-1] + x[-2] = 1 + 0 + 0 = 1$

$$y[1] = x[1] + x[0] + x[-1] = 2 + 1 + 0 = 3$$
$$y[2] = x[2] + x[1] + x[0] = 3 + 2 + 1 = 6$$
$$y[3] = x[3] + x[2] + x[1] = 0 + 3 + 2 = 5$$
$$y[4] = x[4] + x[3] + x[2] = 0 + 0 + 3 = 3$$
$$y[n] = 0 \text{ for } n > 4$$

Method 3 Graphical Method similar to that for CT convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

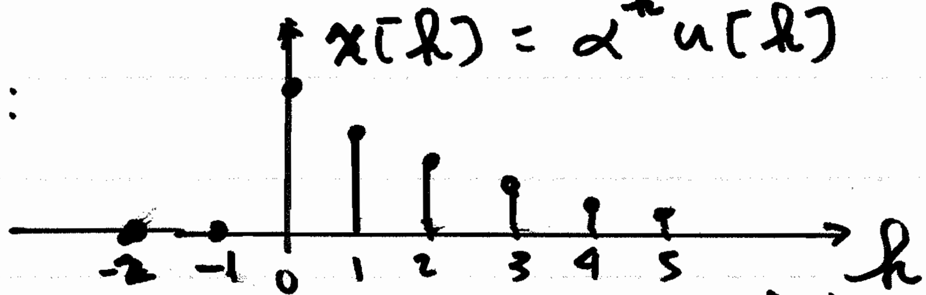
1. View $x[k]$ and $h[-(k-n)]$ as functions of k
2. Flip $h[k]$ about $k=0$ to form $h[-k]$
3. Time-Shift $h[-k]$ to the right by n to form $h[-(k-n)]$
4. Pointwise-multiply to form product $x[k]h[-(k-n)]$
5. Sum the values of the product $x[k]h[-(k-n)]$ over all k
6. ostensibly repeat for each value of n

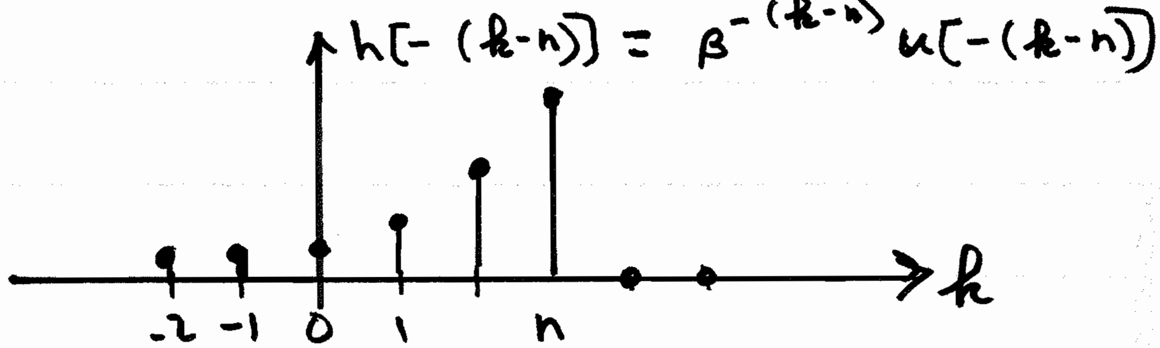
See Example 2.3 in text on pg. 83

• More generally: $x[n] = \alpha^n u[n]$, $h[n] = \beta^n u[n]$ } Hmwk. 3 problem $\alpha \neq \beta$

$y[n] = 0$ for $n < 0$.

• for $n > 0$:





$$y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k$$

$$y[n] = \beta^n \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}} = \beta^n \frac{\beta - \frac{\alpha^n \alpha}{\beta^n}}{\beta - \alpha}$$

$$= \left\{ \frac{\beta}{\beta-\alpha} \beta^n - \frac{\alpha}{\beta-\alpha} \alpha^n \right\} \underbrace{u[n]}_{\text{since starts at } n=0}$$

Example 2.4 in text on pg. 85

$$x[n] = u[n] - u[n-5] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \alpha^n \{u[n] - u[n-7]\}$$

In contrast to text approach (Method 2), this is short enough to do by Method 1

$x[0] = x[1] = x[2] = x[3] = x[4] = 1$

n	0	1	2	3	4	5	6	7	8	9	10	11
$x[0]\,h[n]$	1	α	α^2	α^3	α^4	α^5	α^6	0	0	0	0	0
$x[1]\,h[n-1]$	0	1	α	α^2	α^3	α^4	α^5	α^6	0	0	0	0
$x[2]\,h[n-2]$	0	0	1	α	α^2	α^3	α^4	α^5	α^6	0	0	0
$x[3]\,h[n-3]$	0	0	0	1	α	α^2	α^3	α^4	α^5	α^6	0	0
$x[4]\,h[n-4]$	0	0	0	0	1	α	α^2	α^3	α^4	α^5	α^6	0
$y(n)$	1	$1+\alpha$	$1+\alpha+\alpha^2$							$\alpha^5+\alpha^6$	α^6	0

$$\frac{1-\alpha^3}{1-\alpha}$$

$$\frac{1-\alpha^4}{1-\alpha}$$

$$\frac{1-\alpha^5}{1-\alpha}$$

$$\alpha\,\frac{1-\alpha^5}{1-\alpha}$$

$$\alpha^2\,\frac{(1-\alpha^5)}{1-\alpha}$$

$$\alpha^3\,\frac{(1-\alpha^4)}{1-\alpha}$$

$$\alpha^4\,\frac{(1-\alpha^3)}{1-\alpha}$$

DT convolution satisfies:

1. Commutativity: $x_1[n] * x_2[n] = x_2[n] * x_1[n]$

2. Associativity:

$$\left(x_1[n] * x_2[n]\right) * x_3[n] = x_1[n] * \left(x_2[n] * x_3[n]\right)$$

3. Distributive Property:

$$x_1[n] * \left(x_2[n] + x_3[n]\right)$$
$$= x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

Supplemental Notes on Properties of Convolution which follow from the fact that the Fourier Transform converts convolution in time to multiplication in the frequency domain. (Fourier Transform is 1 to 1 linear operator)

Convolution satisfies distributive property:

$$x[n] * (h_1[n] + h_2[n])$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

$$\xleftrightarrow{\text{DTFT}} X(\omega)\{H_1(\omega) + H_2(\omega)\}$$

$$= X(\omega)H_1(\omega) + X(\omega)H_2(\omega)$$

=> because multiplication satisfies distributive property

Convolution is a commutative operator ⇒ because multiplication is!

$$x[n] * h[n] = h[n] * x[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(\omega)\,H(\omega)$$

$$h[n] * x[n] \xleftrightarrow{\text{DTFT}} H(\omega)\,X(\omega)$$

Convolution satisfies associativity:

$$x[n] * h_1[n] * h_2[n]$$

$$= \big(x[n] * h_1[n]\big) * h_2[n]$$

$$= x[n] * \big(h_1[n] * h_2[n]\big)$$

$$\xleftrightarrow{\text{DTFT}} X(\omega)\, H_1(\omega)\, H_2(\omega)$$

$$X(\omega)\, H_2(\omega)\, H_1(\omega)$$

$$= \big(x[n] * h_2[n]\big) * h_1[n]$$