To prove this property, we use the definition of the Fourier transform in (4.4.1) and differentiate the series term by term with respect to ω. Thus we obtain

$$\begin{aligned}\frac{dX(\omega)}{d\omega} &= \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] \\ &= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n} \\ &= -j\sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}\end{aligned}$$

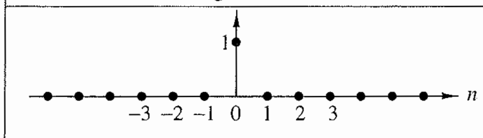
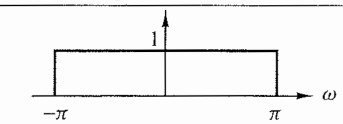
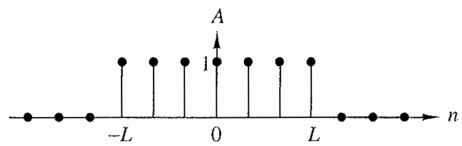
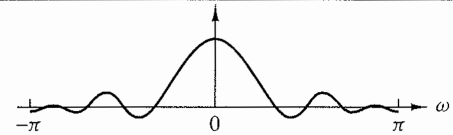
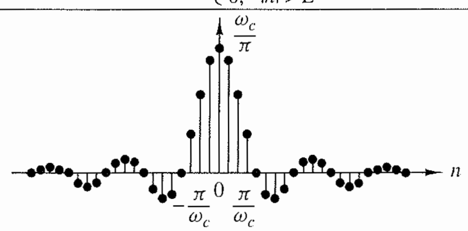
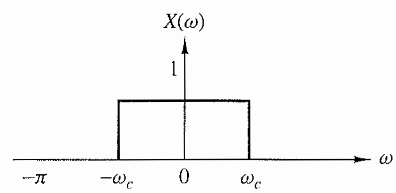
Now we multiply both sides of the equation by j to obtain the desired result in (4.4.58).

The properties derived in this section are summarized in Table 4.5, which serves as a convenient reference. Table 4.6 illustrates some useful Fourier transform pairs that will be encountered in later chapters.

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in		
the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

TABLE 4.6 Some Useful Fourier Transform Pairs for Discrete-Time Aperiodic Signals

Signal $x(n)$	Spectrum $X(\omega)$
 $x(n) = \delta(n)$	 $X(\pi) = 1$
 $x(n) = \begin{cases} A, & \lvert n\rvert \le L \\ 0, & \lvert n\rvert > L \end{cases}$	 $X(\omega) = A\dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \ne 0 \end{cases}$	 $X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \frac{1}{1-ae^{-j\omega}}$

4.5 Summary and References

The Fourier series and the Fourier transform are the mathematical tools for analyzing the characteristics of signals in the frequency domain. The Fourier series is appropriate for representing a periodic signal as a weighted sum of harmonically related sinusoidal components, where the weighting coefficients represent the strengths of each of the harmonics, and the magnitude squared of each weighting coefficient represents the power of the corresponding harmonic. As we have indicated, the Fourier series is one of many possible orthogonal series expansions for a periodic signal. Its importance stems from the characteristic behavior of LTI systems, as we shall see in Chapter 5.

The Fourier transform is appropriate for representing the spectral characteristics of aperiodic signals with finite energy. The important properties of the Fourier transform were also presented in this chapter.

There are many excellent texts on Fourier series and Fourier transforms. For reference, we include the texts by Bracewell (1978), Davis (1963), Dym and McKean (1972), and Papoulis (1962).

where, by definition,

$$x_e(n) = x_R^e(n) + jx_I^e(n) = \frac{1}{2}[x(n) + x^*(-n)]$$

$$x_o(n) = x_R^o(n) + jx_I^o(n) = \frac{1}{2}[x(n) - x^*(-n)]$$

The superscripts e and o denote the even and odd signal components, respectively. We note that $x_e(n) = x_e(-n)$ and $x_o(-n) = -x_o(n)$. From (4.4.36) and the Fourier transform properties established above, we obtain the following relationships:

$$\begin{aligned} x(n) &= [x_R^e(n) + jx_I^e(n)] + [x_R^o(n) + jx_I^o(n)] = x_e(n) + x_o(n) \\ &\qquad\updownarrow \qquad\quad \updownarrow \qquad\qquad\quad \times \qquad\qquad\quad \updownarrow \qquad \updownarrow \\ X(\omega) &= [X_R^e(\omega) + jX_I^e(\omega)] + [X_R^o(\omega) - jX_I^o(\omega)] = X_e(\omega) + X_o(\omega) \end{aligned} \tag{4.4.37}$$

These symmetry properties of the Fourier transform are summarized in Table 4.4 and in Fig 4.4.2. They are often used to simplify Fourier transform calculations in practice.

TABLE 4.4 Symmetry Properties of the Discrete-Time Fourier Transform

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

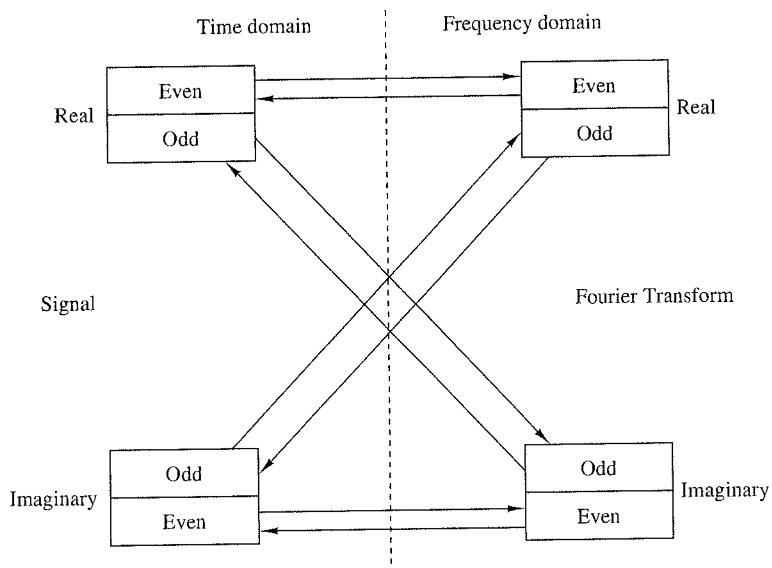


Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

EXAMPLE 4.4.1

Determine and sketch $X_R(\omega)$, $X_I(\omega)$, $|X(\omega)|$, and $\measuredangle X(\omega)$ for the Fourier transform

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \qquad -1 < a < 1 \tag{4.4.38}$$

Solution. By multiplying both the numerator and denominator of (4.4.38) by the complex conjugate of the denominator, we obtain

$$X(\omega) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 - a\cos\omega - ja\sin\omega}{1 - 2a\cos\omega + a^2}$$

This expression can be subdivided into real and imaginary parts. Thus we obtain

$$X_R(\omega) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2}$$

$$X_I(\omega) = -\frac{a\sin\omega}{1 - 2a\cos\omega + a^2}$$

Substitution of the last two equations into (4.4.15) and (4.4.16) yields the magnitude and phase spectra as

$$|X(\omega)| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}} \tag{4.4.39}$$

and

$$\measuredangle X(\omega) = -\tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega} \tag{4.4.40}$$

Figures 4.4.3 and 4.4.4 show the graphical representation of these spectra for $a = 0.8$. The reader can easily verify that as expected, all symmetry properties for the spectra of real signals apply to this case.