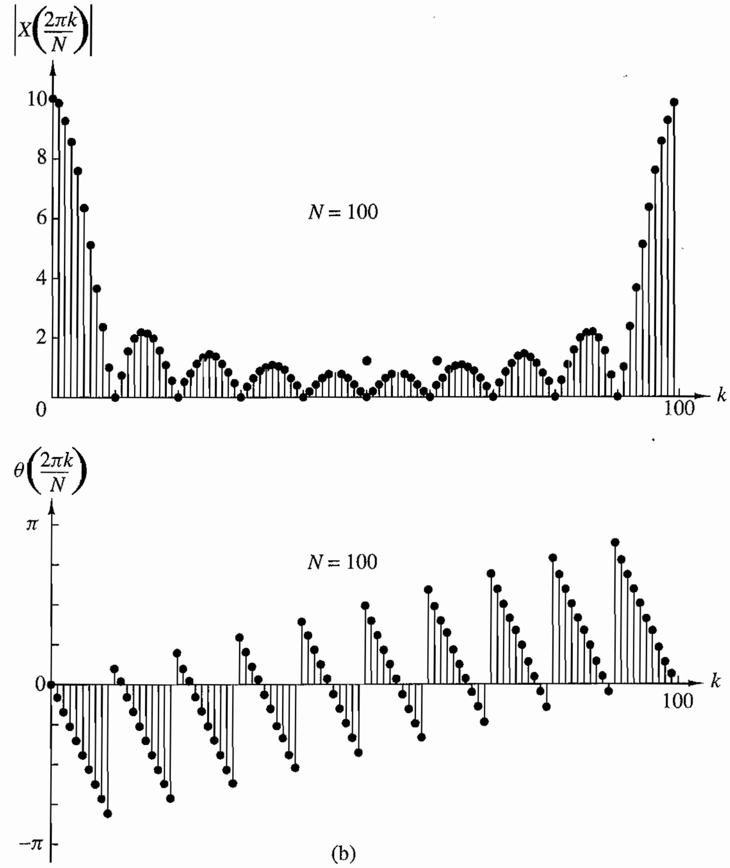


Figure 7.1.6 *continued*

7.1.3 The DFT as a Linear Transformation

The formulas for the DFT and IDFT given by (7.1.18) and (7.1.19) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1 \tag{7.1.22}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1 \tag{7.1.23}$$

where, by definition,

$$W_N = e^{-j2\pi/N} \tag{7.1.24}$$

which is an Nth root of unity.

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and $(N-1)$ complex additions. Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and $N(N-1)$ complex additions.

It is instructive to view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively. Let us define an N-point vector $\mathbf{x}_N$ of the signal sequence $x(n)$, $n = 0, 1, \ldots, N-1$, an N-point vector $\mathbf{X}_N$ of frequency samples, and an $N \times N$ matrix $\mathbf{W}_N$ as

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.1.25}$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{7.1.26}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. We observe that $\mathbf{W}_N$ is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$ exists, then (7.1.26) can be inverted by premultiplying both sides by $\mathbf{W}_N^{-1}$. Thus we obtain

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \tag{7.1.27}$$

But this is just an expression for the IDFT.

In fact, the IDFT as given by (7.1.23) can be expressed in matrix form as

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N \tag{7.1.28}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$. Comparison of (7.1.27) with (7.1.28) leads us to conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^* \tag{7.1.29}$$

which, in turn, implies that

$$\mathbf{W}_N \mathbf{W}_N^* = N \mathbf{I}_N \tag{7.1.30}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix. Therefore, the matrix $\mathbf{W}_N$ in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as $\mathbf{W}_N^*/N$. Of course, the existence of the inverse of $\mathbf{W}_N$ was established previously from our derivation of the IDFT.

EXAMPLE 7.1.3

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Solution. The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$ and then applying the formula (7.1.28).

The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent to the existence of computationally efficient algorithms, known collectively as fast Fourier transform (FFT) algorithms, for computing the DFT and IDFT. This class of algorithms is described in Chapter 8.

7.1.4 Relationship of the DFT to Other Transforms

In this discussion we have indicated that the DFT is an important computational tool for performing frequency analysis of signals on digital signal processors. In view of the other frequency analysis tools and transforms that we have developed, it is important to establish the relationships of the DFT to these other transforms.

Relationship to the Fourier series coefficients of a periodic sequence. A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}, \qquad -\infty < n < \infty \tag{7.1.31}$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \tag{7.1.32}$$

If we compare (7.1.31) and (7.1.32) with (7.1.18) and (7.1.19), we observe that the formula for the Fourier series coefficients has the form of a DFT. In fact, if we define a sequence $x(n) = x_p(n)$, $0 \le n \le N-1$, the DFT of this sequence is simply

$$X(k) = Nc_k \tag{7.1.33}$$

Furthermore, (7.1.31) has the form of an IDFT. Thus the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

Relationship to the Fourier transform of an aperiodic sequence. We have already shown that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \tag{7.1.34}$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{7.1.35}$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.1.36}$$

bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \le N$, in which case

$$x(n) = \hat{x}(n), \qquad 0 \le n \le N-1 \tag{7.1.37}$$

Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship to the z-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{7.1.38}$$

with an ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, $0, 1, 2, \ldots, N-1$, we obtain

$$\begin{aligned} X(k) &\equiv X(z)\big|_{z=e^{j2\pi nk/N}}, \qquad k = 0, 1, \ldots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} \end{aligned} \tag{7.1.39}$$

The expression in (7.1.38) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, which is the topic treated in Section 7.1.1.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT $\{X(k)\}$ as follows:

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n) z^{-n} \\ X(z) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] z^{-n} \\ X(z) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1} \right)^n \\ X(z) &= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}} \end{aligned} \tag{7.1.40}$$

When evaluated on the unit circle, (7.1.40) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}} \tag{7.1.41}$$

This expression for the Fourier transform is a polynomial (Lagrange) interpolation formula for $X(\omega)$ expressed in terms of the values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. With some algebraic manipulations, it is possible to reduce (7.1.41) to the interpolation formula given previously in (7.1.13).

Relationship to the Fourier series coefficients of a continuous-time signal. Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0} \tag{7.1.42}$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N} \end{aligned} \tag{7.1.43}$$

It is clear that (7.1.43) is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \tag{7.1.44}$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN} \tag{7.1.45}$$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

7.2 Properties of the DFT

In Section 7.1.2 we introduced the DFT as a set of N samples $\{X(k)\}$ of the Fourier transform $X(\omega)$ for a finite-duration sequence $\{x(n)\}$ of length $L \leq N$. The sampling of $X(\omega)$ occurs at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$. We demonstrated that the N samples $\{X(k)\}$ uniquely represent the sequence $\{x(n)\}$ in the frequency domain. Recall that the DFT and inverse DFT (IDFT) for an N-point sequence $\{x(n)\}$ are given as

$$\text{DFT:} \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1 \tag{7.2.1}$$

$$\text{IDFT:} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1 \tag{7.2.2}$$

where W_N is defined as

$$W_N = e^{-j2\pi/N} \tag{7.2.3}$$

In this section we present the important properties of the DFT. In view of the relationships established in Section 7.1.4 between the DFT and Fourier series, and Fourier transforms and z-transforms of discrete-time signals, we expect the properties of the DFT to resemble the properties of these other transforms and series. However, some important differences exist, one of which is the circular convolution property derived in the following section. A good understanding of these properties is extremely helpful in the application of the DFT to practical problems.

The notation used below to denote the N-point DFT pair $x(n)$ and $X(k)$ is

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

7.2.1 Periodicity, Linearity, and Symmetry Properties

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n \tag{7.2.4}$$

$$X(k+N) = X(k) \qquad \text{for all } k \tag{7.2.5}$$

These periodicities in $x(n)$ and $X(k)$ follow immediately from formulas (7.2.1) and (7.2.2) for the DFT and IDFT, respectively.

We previously illustrated the periodicity property in the sequence $x(n)$ for a given DFT. However, we had not previously viewed the DFT $X(k)$ as a periodic sequence. In some applications it is advantageous to do this.

Linearity. If

$$x_1(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \tag{7.2.6}$$

This property follows immediately from the definition of the DFT given by (7.2.1).

Circular Symmetries of a Sequence. As we have seen, the N-point DFT of a finite duration sequence $x(n)$, of length $L \le N$, is equivalent to the N-point DFT of a periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending $x(n)$, that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{7.2.7}$$

Now suppose that we shift the periodic sequence $x_p(n)$ by k units to the right. Thus we obtain another periodic sequence

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN) \tag{7.2.8}$$

The finite-duration sequence

$$x'(n) = \begin{cases} x'_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.2.9}$$

is related to the original sequence $x(n)$ by a circular shift. This relationship is illustrated in Fig. 7.2.1 for $N = 4$.

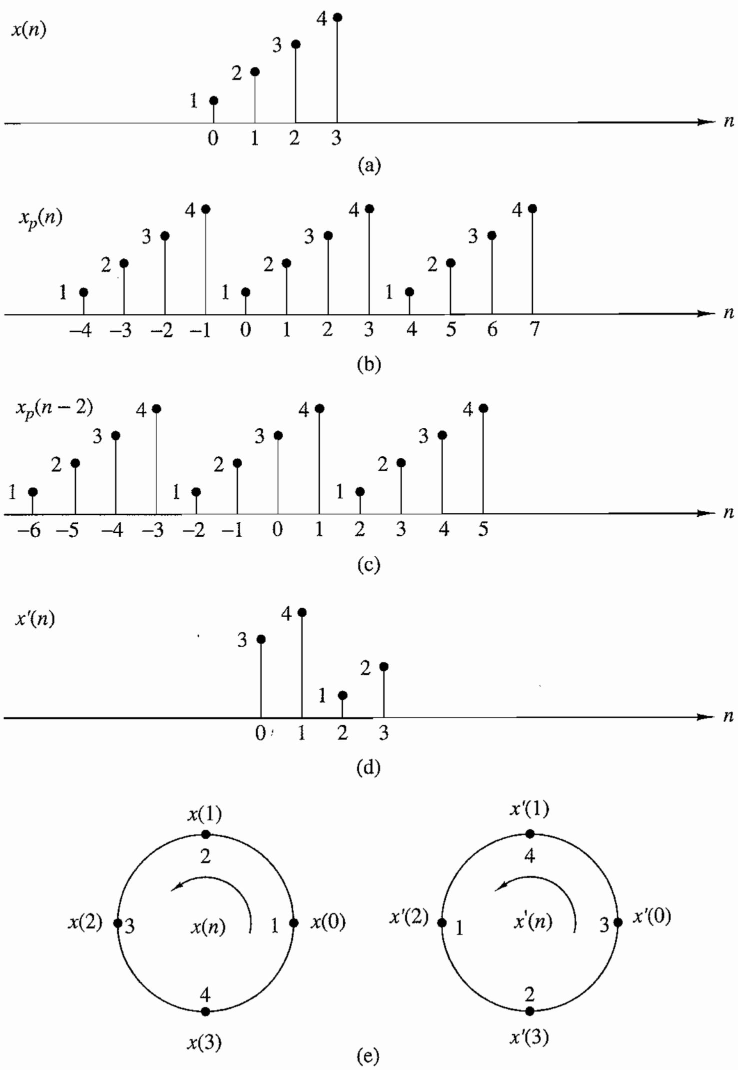


Figure 7.2.1 Circular shift of a sequence.

In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$\begin{aligned} x'(n) &= x(n-k, \text{modulo } N) \\ &\equiv x((n-k))_N \end{aligned} \tag{7.2.10}$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n-2))_4$$

which implies that

$$\begin{aligned} x'(0) &= x((-2))_4 = x(2) \\ x'(1) &= x((-1))_4 = x(3) \\ x'(2) &= x((0))_4 = x(0) \\ x'(3) &= x((1))_4 = x(1) \end{aligned}$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counterclockwise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension, and vice versa.

The inherent periodicity resulting from the arrangement of the N-point sequence on the circumference of a circle dictates a different definition of even and odd symmetry, and time reversal of a sequence.

An N-point sequence is called circularly *even* if it is symmetric about the point zero on the circle. This implies that

$$x(N-n) = x(n), \qquad 1 \le n \le N-1 \tag{7.2.11}$$

An N-point sequence is called circularly *odd* if it is antisymmetric about the point zero on the circle. This implies that

$$x(N-n) = -x(n), \qquad 1 \le n \le N-1 \tag{7.2.12}$$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is simply given as

$$x((-n))_N = x(N-n), \qquad 0 \le n \le N-1 \tag{7.2.13}$$

This time reversal is equivalent to plotting $x(n)$ in a clockwise direction on a circle.

An equivalent definition of even and odd sequences for the associated periodic sequence $x_p(n)$ is given as follows

$$\begin{aligned} \text{even:} \quad & x_p(n) = x_p(-n) = x_p(N-n) \\ \text{odd:} \quad & x_p(n) = -x_p(-n) = -x_p(N-n) \end{aligned} \tag{7.2.14}$$

If the periodic sequence is complex valued, we have

$$\begin{aligned} \text{conjugate even:} \quad & x_p(n) = x_p^*(N-n) \\ \text{conjugate odd:} \quad & x_p(n) = -x_p^*(N-n) \end{aligned} \tag{7.2.15}$$

These relationships suggest that we decompose the sequence $x_p(n)$ as

$$x_p(n) = x_{pe}(n) + x_{po}(n) \tag{7.2.16}$$

where

$$\begin{aligned} x_{pe}(n) &= \frac{1}{2}[x_p(n) + x_p^*(N-n)] \\ x_{po}(n) &= \frac{1}{2}[x_p(n) - x_p^*(N-n)] \end{aligned} \tag{7.2.17}$$

Symmetry properties of the DFT. The symmetry properties for the DFT can be obtained by applying the methodology previously used for the Fourier transform. Let us assume that the N-point sequence $x(n)$ and its DFT are both complex valued. Then the sequences can be expressed as

$$x(n) = x_R(n) + jx_I(n), \qquad 0 \le n \le N-1 \tag{7.2.18}$$

$$X(k) = X_R(k) + jX_I(k), \qquad 0 \le k \le N-1 \tag{7.2.19}$$

By substituting (7.2.18) into the expression for the DFT given by (7.2.1), we obtain

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right] \tag{7.2.20}$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right] \tag{7.2.21}$$

Similarly, by substituting (7.2.19) into the expression for the IDFT given by (7.2.2), we obtain

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi kn}{N} - X_I(k) \sin \frac{2\pi kn}{N} \right] \tag{7.2.22}$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi kn}{N} + X_I(k) \cos \frac{2\pi kn}{N} \right] \tag{7.2.23}$$

Real-valued sequences. If the sequence $x(n)$ is real, it follows directly from (7.2.1) that

$$X(N-k) = X^*(k) = X(-k) \tag{7.2.24}$$

Consequently, $|X(N-k)| = |X(k)|$ and $\angle X(N-k) = -\angle X(k)$. Furthermore, $x_I(n) = 0$ and therefore $x(n)$ can be determined from (7.2.22), which is another form for the IDFT.

Real and even sequences. If $x(n)$ is real and even, that is,

$$x(n) = x(N-n), \qquad 0 \le n \le N-1$$

then (7.2.21) yields $X_I(k) = 0$. Hence the DFT reduces to

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N}, \qquad 0 \le k \le N-1 \tag{7.2.25}$$

which is itself real valued and even. Furthermore, since $X_I(k) = 0$, the IDFT reduces to

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N}, \qquad 0 \le n \le N-1 \tag{7.2.26}$$

Real and odd sequences. If $x(n)$ is real and odd, that is,

$$x(n) = -x(N-n), \qquad 0 \le n \le N-1$$

then (7.2.20) yields $X_R(k) = 0$. Hence

$$X(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi kn}{N}, \qquad 0 \le k \le N-1 \tag{7.2.27}$$

which is purely imaginary and odd. Since $X_R(k) = 0$, the IDFT reduces to

$$x(n) = j\frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}, \qquad 0 \le n \le N-1 \tag{7.2.28}$$

Purely imaginary sequences. In this case, $x(n) = jx_I(n)$. Consequently, (7.2.20) and (7.2.21) reduce to

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N} \tag{7.2.29}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N} \tag{7.2.30}$$

We observe that $X_R(k)$ is odd and $X_I(k)$ is even.

If $x_I(n)$ is odd, then $X_I(k) = 0$ and hence $X(k)$ is purely real. On the other hand, if $x_I(n)$ is even, then $X_R(k) = 0$ and hence $X(k)$ is purely imaginary.

The symmetry properties given above may be summarized as follows:

$$\begin{array}{c} x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n) \\ \updownarrow \\ X(k) = X_R^e(k) + X_R^o(k) + jX_I^e(k) + jX_I^o(k) \end{array} \tag{7.2.31}$$

All the symmetry properties of the DFT can easily be deduced from (7.2.31). For example, the DFT of the sequence

$$x_{pe}(n) = \tfrac{1}{2}[x_p(n) + x_p^*(N-n)]$$

is

$$X_R(k) = X_R^e(k) + X_R^o(k)$$

The symmetry properties of the DFT are summarized in Table 7.1. Exploitation of these properties for the efficient computation of the DFT of special sequences is considered in some of the problems at the end of the chapter.

TABLE 7.1 Symmetry Properties of the DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$
$x(n)$	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\lvert X(k)\rvert = \lvert X(N-k)\rvert$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

7.2.2 Multiplication of Two DFTs and Circular Convolution

Suppose that we have two finite-duration sequences of length N, $x_1(n)$ and $x_2(n)$. Their respective N-point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \tag{7.2.32}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \tag{7.2.33}$$

If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length N. Let us determine the relationship between $x_3(n)$ and the sequences $x_1(n)$ and $x_2(n)$.

We have

$$X_3(k) = X_1(k)X_2(k), \qquad k = 0, 1, \ldots, N-1 \tag{7.2.34}$$

The IDFT of $\{X_3(k)\}$ is

$$\begin{aligned} x_3(m) &= \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi km/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)e^{j2\pi km/N} \end{aligned} \tag{7.2.35}$$

Suppose that we substitute for $X_1(k)$ and $X_2(k)$ in (7.2.35) using the DFTs given in (7.2.32) and (7.2.33). Thus we obtain

$$\begin{aligned} x_3(m) &= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)e^{-j2\pi kl/N}\right]e^{j2\pi km/N} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \end{aligned} \tag{7.2.36}$$

The inner sum in the brackets in (7.2.36) has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \tag{7.2.37}$$

where a is defined as

$$a = e^{j2\pi(m-n-l)/N}$$

We observe that $a = 1$ when $m - n - l$ is a multiple of N. On the other hand, $a^N = 1$ for any value of $a \neq 0$. Consequently, (7.2.37) reduces to

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m - n + pN = ((m-n))_N, \quad p \text{ an integer} \\ 0, & \text{otherwise} \end{cases} \tag{7.2.38}$$

If we substitute the result in (7.2.38) into (7.2.36), we obtain the desired expression for $x_3(m)$ in the form

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \qquad m = 0, 1, \ldots, N-1 \tag{7.2.39}$$

The expression in (7.2.39) has the form of a convolution sum. However, it is not the ordinary linear convolution that was introduced in Chapter 2, which relates the output sequence $y(n)$ of a linear system to the input sequence $x(n)$ and the impulse response $h(n)$. Instead, the convolution sum in (7.2.39) involves the index $((m-n))_N$ and is called *circular convolution.* Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

The following example illustrates the operations involved in circular convolution.

EXAMPLE 7.2.1

Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Solution. Each sequence consists of four nonzero points. For the purposes of illustrating the operations involved in circular convolution, it is desirable to graph each sequence as points on a circle. Thus the sequences $x_1(n)$ and $x_2(n)$ are graphed as illustrated in Fig. 7.2.2(a). We note that the sequences are graphed in a counterclockwise direction on a circle. This establishes the reference direction in rotating one of the sequences relative to the other.

Now, $x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as specified by (7.2.39). Beginning with $m = 0$ we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))N$$

$x_2((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as illustrated in Fig. 7.2.2(b). In other words, the folded sequence is simply $x_2(n)$ graphed in a clockwise direction.

The product sequence is obtained by multiplying $x_1(n)$ with $x_2((-n))_4$, point by point. This sequence is also illustrated in Fig. 7.2.2(b). Finally, we sum the values in the product sequence to obtain

$$x_3(0) = 14$$

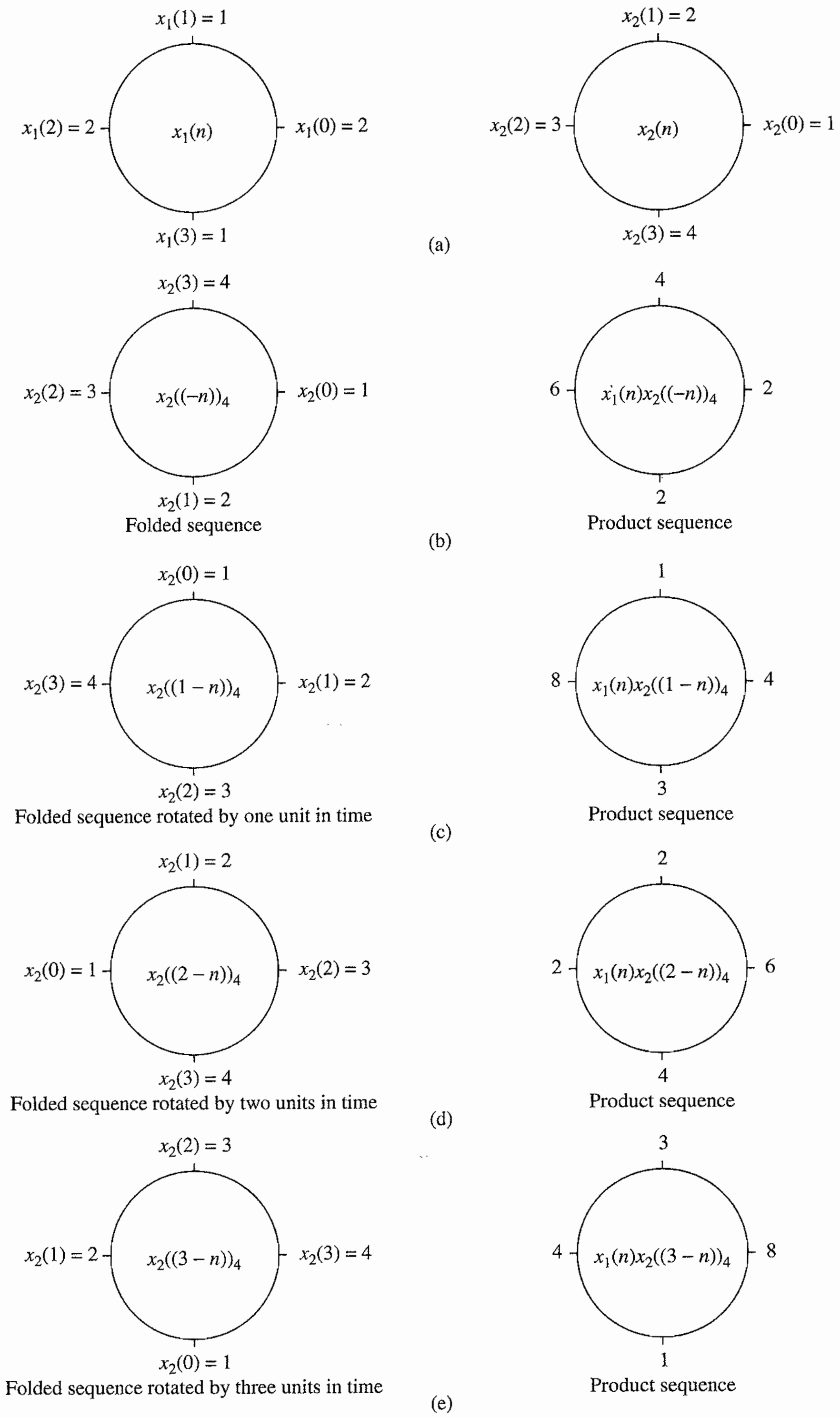


Figure 7.2.2 Circular convolution of two sequences.

For $m = 1$ we have

$$x_3(1) = \sum_{n=0}^{3} x_1(n)x_2((1-n))_4$$

It is easily verified that $x_2((1-n))_4$ is simply the sequence $x_2((-n))_4$ rotated counterclockwise by one unit in time as illustrated in Fig. 7.2.2(c). This rotated sequence multiplies $x_1(n)$ to yield the product sequence, also illustrated in Fig. 7.2.2(c). Finally, we sum the values in the product sequence to obtain $x_3(1)$. Thus

$$x_3(1) = 16$$

For $m = 2$ we have

$$x_3(2) = \sum_{n=0}^{3} x_1(n)x_2((2-n))_4$$

Now $x_2((2-n))_4$ is the folded sequence in Fig. 7.2.2(b) rotated two units of time in the counterclockwise direction. The resultant sequence is illustrated in Fig. 7.2.2(d) along with the product sequence $x_1(n)x_2((2-n))_4$. By summing the four terms in the product sequence, we obtain

$$x_3(2) = 14$$

For $m = 3$ we have

$$x_3(3) = \sum_{n=0}^{3} x_1(n)x_2((3-n))_4$$

The folded sequence $x_2((-n))_4$ is now rotated by three units in time to yield $x_2((3-n))_4$ and the resultant sequence is multiplied by $x_1(n)$ to yield the product sequence as illustrated in Fig. 7.2.2(e). The sum of the values in the product sequence is

$$x_3(3) = 16$$

We observe that if the computation above is continued beyond $m = 3$, we simply repeat the sequence of four values obtained above. Therefore, the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$ yields the sequence

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

From this example, we observe that circular convolution involves basically the same four steps as the ordinary *linear convolution* introduced in Chapter 2: *folding* (time reversing) one sequence, *shifting* the folded sequence, *multiplying* the two sequences to obtain a product sequence, and finally, *summing* the values of the product sequence. The basic difference between these two types of convolution is that, in circular convolution, the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences modulo N. In linear convolution, there is no modulo N operation.

The reader can easily show from our previous development that either one of the two sequences may be folded and rotated without changing the result of the circular convolution. Thus

$$x_3(m) = \sum_{n=0}^{N-1} x_2(n)x_1((m-n))_N, \qquad m = 0, 1, \ldots, N-1 \tag{7.2.40}$$

The following example serves to illustrate the computation of $x_3(n)$ by means of the DFT and IDFT.

EXAMPLE 7.2.2

By means of the DFT and IDFT, determine the sequence $x_3(n)$ corresponding to the circular convolution of the sequences $x_1(n)$ and $x_2(n)$ given in Example 7.2.1.

Solution. First we compute the DFTs of $x_1(n)$ and $x_2(n)$. The four-point DFT of $x_1(n)$ is

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2}$$

Thus

$$X_1(0) = 6, \qquad X_1(1) = 0, \qquad X_1(2) = 2, \qquad X_1(3) = 0$$

The DFT of $x_2(n)$ is

$$X_2(k) = \sum_{n=0}^{3} x_2(n)e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$$

Thus

$$X_2(0) = 10, \qquad X_2(1) = -2 + j2, \qquad X_2(2) = -2, \qquad X_2(3) = -2 - j2$$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k)X_2(k)$$

or, equivalently,

$$X_3(0) = 60, \qquad X_3(1) = 0, \qquad X_3(2) = -4, \qquad X_3(3) = 0$$

Now, the IDFT of $X_3(k)$ is

$$x_3(n) = \sum_{k=0}^{3} X_3(k)e^{j2\pi nk/4}, \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4}(60 - 4e^{j\pi n})$$

Thus

$$x_3(0) = 14, \qquad x_3(1) = 16, \qquad x_3(2) = 14, \qquad x_3(3) = 16$$

which is the result obtained in Example 7.2.1 from circular convolution.

We conclude this section by formally stating this important property of the DFT.

Circular convolution. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \; \textcircled{\scriptsize N} \; x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)X_2(k) \tag{7.2.41}$$

where $x_1(n) \; \textcircled{\scriptsize N} \; x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$.

7.2.3 Additional DFT Properties

Time reversal of a sequence. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k) \tag{7.2.42}$$

Hence reversing the N-point sequence in time is equivalent to reversing the DFT values. Time reversal of a sequence $x(n)$ is illustrated in Fig. 7.2.3.

Proof From the definition of the DFT in (7.2.1) we have

$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi kn/N}$$

If we change the index from n to $m = N - n$, then

$$\text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N}$$

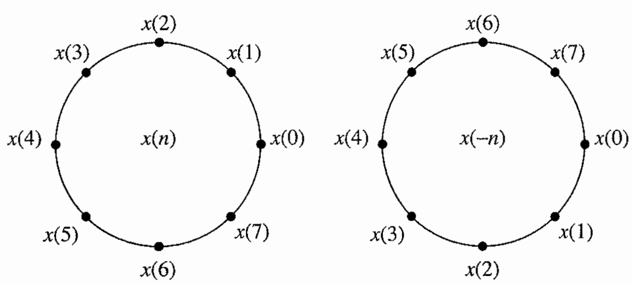


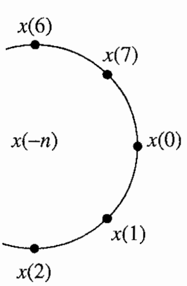
Figure 7.2.3 Time reversal of a sequence.

(7.2.41)

nce $x_1(n)$ and

(7.2.42)

rsing the DFT



$$= \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N} = X(N-k)$$

We note that $X(N-k) = X((-k))_N$, $0 \le k \le N-1$.

Circular time shift of a sequence. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)e^{-j2\pi kl/N} \tag{7.2.43}$$

Proof From the definition of the DFT we have

$$\text{DFT}\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$+ \sum_{n=l}^{N-1} x(n-l)e^{-j\pi kn/N}$$

But $x((n-l))_N = x(N-l+n)$. Consequently,

$$\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N-l+n)e^{-j2\pi kn/N}$$

$$= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$$

Furthermore,

$$\sum_{n=l}^{N-1} x(n-l)e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)e^{-j2\pi k(m+l)/N}$$

Therefore,

$$\text{DFT}\{x((n-l))\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$$

$$= X(k)e^{-j2\pi kl/N}$$

Circular frequency shift. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x(n)e^{j2\pi ln/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((k-l))_N \tag{7.2.44}$$

Hence, the multiplication of the sequence $x(n)$ with the complex exponential sequence $e^{j2\pi kn/N}$ is equivalent to the circular shift of the DFT by l units in frequency. This is the dual to the circular time-shifting property and its proof is similar to that of the latter.

Complex-conjugate properties. If

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x^*(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k) \tag{7.2.45}$$

The proof of this property is left as an exercise for the reader. The IDFT of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

Therefore,

$$x^*((-n))_N = x^*(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*(k) \tag{7.2.46}$$

Circular correlation. In general, for complex-valued sequences $x(n)$ and $y(n)$, if

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

and

$$y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$$

then

$$\tilde{r}_{xy}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k) \tag{7.2.47}$$

where $\tilde{r}_{xy}(l)$ is the (unnormalized) circular crosscorrelation sequence, defined as

$$\tilde{r}xy(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))N$$

Proof We can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$, that is,

$$\tilde{r}_{xy}(l) = x(l) \textcircled{N} y^*(-l)$$

Then, with the aid of the properties in (7.2.41) and (7.2.46), the N-point DFT of $\tilde{r}_{xy}(l)$ is

$$\tilde{R}_{xy}(k) = X(k)Y^*(k)$$

In the special case where $y(n) = x(n)$, we have the corresponding expression for the circular autocorrelation of $x(n)$,

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2 \tag{7.2.48}$$

Multiplication of two sequences. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n)x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \frac{1}{N} X_1(k) \textcircled{N} X_2(k) \tag{7.2.49}$$

This property is the dual of (7.2.41). Its proof follows simply by interchanging the roles of time and frequency in the expression for the circular convolution of two sequences.

Parseval's Theorem. For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

and

$$y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \tag{7.2.50}$$

Proof The property follows immediately from the circular correlation property in (7.2.47). We have

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \tilde{r}_{xy}(0)$$

TABLE 7.2 Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \textcircled{N} X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{R}_{xy}(k)e^{j2\pi kl/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)e^{j2\pi kl/N}$$

Hence (7.2.50) follows by evaluating the IDFT at $l = 0$.

The expression in (7.2.50) is the general form of Parseval's theorem. In the special case where $y(n) = x(n)$, (7.2.50) reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \tag{7.2.51}$$

which expresses the energy in the finite-duration sequence $x(n)$ in terms of the frequency components $\{X(k)\}$.

The properties of the DFT given above are summarized in Table 7.2.

7.3 Linear Filtering Methods Based on the DFT

Since the DFT provides a discrete frequency representation of a finite-duration sequence in the frequency domain, it is interesting to explore its use as a computational tool for linear system analysis and, especially, for linear filtering. We have already established that a system with frequency response $H(\omega)$, when excited with an input