- Some observations re: the N-pt DFT of a sinewave of length N with a frequency of the form $\omega = 2\pi k/N$, where $k \in \{0, 1, \ldots, N-1\}$, i.e., that means the frequency of the sinewave is one of the N frequencies we sample the DTFT at. Leads to "nice" result:

(1)

$$e^{j 2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\} \overset{DFT}{\underset{N}{\longleftrightarrow}} N\,\delta[k-k_0]$$

$\Rightarrow$ sample at peak of "sinc" function and at all the zero crossings in $[0, 2\pi)$

2

- Consider $h[n]$ of length $M < N$

$$h[n] \underset{N}{\overset{DFT}{\longleftrightarrow}} H_N(k)$$

- let $x[n] = e^{j 2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\}$

$$\underset{N}{\overset{DFT}{\longleftrightarrow}} X_N(k) = N\,\delta[k-k_0]$$

- Consider:

$$y_p[n] = ? \underset{N}{\overset{DFT}{\longleftrightarrow}} Y_N(k) = X_N(k)\,H_N(k)$$

The answer is easy since:

$$Y_N(k) = N\,\delta[k-k_0]\,H_N(k)$$
$$= N\,H_N(k_0)\,\delta[k-k_0]$$

• Thus, the N-pt inverse DFT is: (3)

$$y_N[n] = \quad \xleftrightarrow[N]{DFT} \; N\,H_N(k_0)\,\delta[k-k_0]$$

$$= H_N(k_0)\, e^{j 2\pi \frac{k_0}{N} n} \underbrace{\{u[n] - u[n-N]\}}$$

• THUS:

$$y_N[n] = e^{j 2\pi \frac{k_0}{N} n} \underbrace{\; \text{Ⓝ} \;}_{\text{circular convolution}} h[n] = H_N(k_0)\, e^{j 2\pi \frac{k_0}{N} n}$$

$$\boxed{H_N(k_0) = H(\omega)\big|_{\omega = 2\pi \frac{k_0}{N}}}$$

This mimicks what happens with an infinite-length sinewave and linear convolution:

$$y[n] = e^{j\omega_0 n} * h[n] = H(\omega_0)\, e^{j\omega_0 n}$$

- Since the sineware $x[n]$ is of length N and $h[n]$ is of length $M\ (<N)$, then the N-pt DFT based processing will lead to time-domain aliasing (4)
- the linear convolution is of length $N+M-1$
- what happens is that the $M-1$ "transient" points at the end of the linear convolution are aliased into the $M-1$ "transient" points at the beginning of the linear convolution AND they amazingly yield $M-1$ "good" points

$\Rightarrow$ yielding, in turn, N values of the sineware

- See Problem 1 from Final Exam from Fall 2009 : (5)

- In convolving $x[n] = e^{j 2\pi \frac{k_0}{N} n} \{u[n] - u[n-N]\}$ with causal $h[n]$ of length $M < N$, consider "flipping" around $x[n]$ while doing graphical convolution to determine the limits on the sum

- $y[n] = \sum_{\tilde{k}=?}^{?} h[\tilde{k}]\, x[n-\tilde{k}]$

- For $0 \leq n \leq$ M-2 : partial overlap

$$y[n] = \sum_{\tilde{k}=0}^{n} h[\tilde{k}]\, e^{j 2\pi \frac{k_0}{N} (n-\tilde{k})}$$

$$y[n] = \left\{ \sum_{\tilde{k}=0}^{n} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N} \tilde{k}} \right\} e^{j 2\pi \frac{k_0}{N} n} \qquad (6)$$

- Next, consider $M-1$ "transient" points corresponding to the partial overlap points at the end
- For $N \le n \le N+M-2$:

$$y[n] = \sum_{\tilde{k}=n-N+1}^{M-1} h[\tilde{k}]\, e^{j 2\pi \frac{k_0}{N} (n-\tilde{k})}$$

$$= \left\{ \sum_{\tilde{k}=n-N+1}^{M-1} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N} \tilde{k}} \right\} e^{j 2\pi \frac{k_0}{N} n}$$

• Now, the time-domain aliasing formula tells us that for the first M-1 pts. of the circular convolution (DFT processing) (7)

$$y_N[n] = y[n] + y[n+N] \qquad n = 0, 1, \ldots M-2$$

• Examine 2^{nd} term $y[n+N]$ for $n = 0, 1, \ldots, M-2$

$$y[n+N] = \left\{ \sum_{\tilde{k}=(n+N)-N+1}^{M-1} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N} \tilde{k}} \right\} e^{j 2\pi \frac{k_0}{N}(n+N)}$$

$$= \left\{ \sum_{\tilde{k}=n+1}^{M-1} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N} \tilde{k}} \right\} e^{j 2\pi \frac{k_0}{N} n}$$

(8)

- Combine that with the previous expression for $y(n)$, for $n=0,1,\ldots,M-2$

$$y_N[n] = \left\{\sum_{\tilde{k}=0}^{M-1} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N}\tilde{k}}\right\} e^{j 2\pi \frac{k_0}{N} n}$$

$$= \left\{\sum_{n=0}^{M-1} h[n]\, e^{-j 2\pi \frac{k_0}{N} n}\right\} e^{j 2\pi \frac{k_0}{N} n}$$

$$= H_N(k_0)\, e^{j 2\pi \frac{k_0}{N} n} \Rightarrow \text{sinewave!}$$

$n=0,1,\ldots,M-1$

· Combine this with the result for the full overlap region $M \le n \le N-1$:

$$y_N[n] = \sum_{\tilde{k}=0}^{M-1} h[\tilde{k}]\, e^{j 2\pi \frac{k_0}{N}(n-k)} \qquad \text{(a)}$$

· and we get all N-pts. of pure sinewave since again

$$y[n] = \left\{ \sum_{\tilde{k}=0}^{M-1} h[\tilde{k}]\, e^{-j 2\pi \frac{k_0}{N} \tilde{k}} \right\} e^{j 2\pi \frac{k_0}{N} n}$$

$$= \left\{ \sum_{n=0}^{M-1} h[n]\, e^{-j 2\pi \frac{k_0}{N} n} \right\} e^{j 2\pi \frac{k_0}{N} n}$$

- Notes: the DFT based processing is linear

$$\sum_{l=0}^{N-1} \alpha_l e^{j 2\pi \frac{l}{N} n} \{u[n] - u[n-N]\} \underset{N}{\overset{DFT}{\longleftrightarrow}} X_N(k)$$

$$h[n] \underset{N}{\overset{DFT}{\longleftrightarrow}} H_N(k)$$

$$y_N[n] = \quad \underset{N}{\overset{DFT}{\longleftrightarrow}} Y_N(k) = H_N(k) X_N(k)$$

$$= \sum_{l=0}^{N-1} \alpha_l H_N(l) e^{j 2\pi \frac{l}{N} n} \{u[n] - u[n-N]\}$$

- Recall, from Euler's formula: (11)

$$\cos\left(2\pi \frac{k_0}{N} n\right) = \frac{1}{2} e^{j 2\pi \frac{k_0}{N} n} + \frac{1}{2} e^{-j 2\pi \frac{k_0}{N} n}$$

$$= \frac{1}{2} e^{j 2\pi \frac{k_0}{N} n} + \frac{1}{2} e^{j 2\pi \frac{(N-k_0)}{N} n}$$

- this is important since for all the previous results (previous pages), the integer k_0 in $e^{j 2\pi \frac{k_0}{N} n}$ was between 0 and $N-1$ (i.e., not negative)