N- pt DFT Matrix

DFT = Discrete Fourier Transform

$$\underline{W}_N = \begin{bmatrix} \left\{e^{-j0n}\left(u[n]-u[n-N]\right)\right\} \\ \left\{e^{-j\frac{2\pi}{N}n}\left(u[n]-u[n-N]\right)\right\} \\ \left\{e^{-j\frac{4\pi}{N}n}\left(u[n]-u[n-N]\right)\right\} \\ \vdots \\ \left\{e^{-j\frac{(N-1)2\pi}{N}}\left(u[n]-u[n-N]\right)\right\} \end{bmatrix} = \begin{bmatrix} \underline{s}^H(0) \\ \underline{s}^H\left(\frac{2\pi}{N}\right) \\ \underline{s}^H\left(\frac{4\pi}{N}\right) \\ \vdots \\ \underline{s}^H\left(\frac{(N-1)2\pi}{N}\right) \end{bmatrix}$$

where: $\underset{N\times 1}{\underline{s}}(\omega) = \left[1 \;\; e^{j\omega} \;\; e^{j2\omega} \;\; \cdots \;\; e^{j(N-1)\omega}\right]^T$

Notation: superscript H = conjugate-transpose

$\underline{A}^H = \underline{A}^{*T}$ H stands for Hermitian Transpose

In Matlab, it's the tic mark A'

(For transpose only in Matlab, use A.')

- The Inverse DFT Matrix is : $\underline{W}_N^{-1}$
- Since the N-length sinewaves comprising the rows of $\underline{W}_N$ are orthogonal (proved previously) it follows that:

$$\underline{W}_N^{-1} = \frac{1}{N}\underline{W}_N^H \quad \left(\text{since: } \underline{W}_N^H \underline{W}_N = N\,\underline{I}_N\right)$$

- Each <u>column</u> of $\underline{W}_N^{-1} = \frac{1}{N}\underline{W}_N^H$ is a sinewave of length N

(3)

- $\underline{W}_N^{-1} = \frac{1}{N} \underline{W}_N^H$

$$= \frac{1}{N}\left[\underline{s}(0) \;\vdots\; \underline{s}\left(\frac{2\pi}{N}\right) \;\vdots\; \underline{s}\left(\frac{4\pi}{N}\right) \;\vdots\; \cdots \;\vdots\; \underline{s}\left(\frac{(N-1)2\pi}{N}\right)\right]$$

- each column is an $N \times 1$ vector containing a sinewave at frequency $k\frac{2\pi}{N}$ of length N, where $k = 0, 1, \ldots, N-1$
- $\Rightarrow$ N equi-spaced frequencies between 0 and 2π (not including 2π)

- Partitioned form of a matrix-vector product:

- Consider an $m \times n$ matrix $\underline{A}$ which has n columns of dimension $m \times 1$ denoted $\underline{a}_k$

$k = 1, \ldots, n$

$$\underline{A} = \left[\, \underline{a}_1 \;\vdots\; \underline{a}_2 \;\vdots\; \cdots \;\vdots\; \underline{a}_n \,\right]$$

- Consider post-multiplying $\underline{A}$ by the $n \times 1$ vector $\underset{n\times 1}{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\Rightarrow$ $\underset{m\times 1}{\underline{y}} = \underset{m\times n}{\underline{A}}\; \underset{n\times 1}{\underline{x}}$

- can easily see that:

$$\underline{y} = \underline{A}\,\underline{x} = \underline{a}_1 x_1 + \underline{a}_2 x_2 + \ldots + \underline{a}_n x_n$$

= linear combination of the columns of $\underline{A}$

$$= \sum_{k=1}^{n} \underline{a}_k x_k$$

• In context of OFDM application, let $\underline{b}$ be an $N \times 1$ vector containing the N information symbols to be transmitted ⇒ as the amplitudes of N sinewaves ⇒

•
$$\underset{N\times 1}{\underline{x}} = \frac{1}{N} \underset{N\times N}{\underline{W}_N^H} \underset{N\times 1}{\underline{b}} = \frac{1}{N} \sum_{k=0}^{N-1} \underline{s}\left(k\frac{2\pi}{N}\right) b_k$$

or $b[k]$

= sum of sinewaves where the information symbols are the respective amplitudes

⇒ for simple BPSK: $b_k = 1$ representing a "1"

or $b_k = -1$ representing a "0"

- $\underline{W}_N^{-1} = \frac{1}{N}\underline{W}_N^H$ contains N orthogonal sinewaves of length N (as columns) where each sinewaves is periodic with period N
- For any frequency $k\frac{2\pi}{N} = 2\pi\frac{k}{N}$ for which k and N have common divisors, the sinewaves repeat more quickly (see notes from 1st week of class)
- The $N \times N$ matrix $\underline{W}_N^{-1} = \frac{1}{N}\underline{W}_N^H$ contains N^2 numbers but only N unique numbers which are equi-spaced around the unit circle

$$e^{j\frac{k2\pi}{N}}, \quad k = 0, 1, \ldots, N-1$$

- See matlab demos DFT_MatrixColors.m and dftcolor.m

- The Inverse FFT (Fast Fourier Transform) exploits all of these features to reduce the computational complexity of forming

$$\underset{N\times 1}{\underline{x}} = \frac{1}{N}\underset{N\times N}{\underline{W}_N^H}\ \underset{N\times 1}{\underline{b}}$$

from N^2 multiplications to $\frac{N}{2}\log_2(N)$ multiplications

for $N=2^v$, v integer

- FFT's and Inverse FFT's are to be developed in the next lecture
- In the context of OFDM, the Inverse FFT is the computationally efficient mechanism by which the sum of sinewaves is formed where the respective amplitudes of the sinewaves = Info Symbols

- For OFDM, at the receiver, we exploit the orthogonality of the sinewaves to form

$$\underline{z} = \underline{W}_N \underline{y} \quad \text{where: } \underline{y} = \frac{1}{N} \underline{W}_N^H \underline{\mathcal{H}} \underline{b}$$

$\underline{\mathcal{H}}$ is an $N \times N$ diagonal matrix with the channel gains:

$$h[n] \xleftrightarrow{\text{DTFT}} H(\omega)$$

multipath channel impulse response — multipath channel frequency response

$$\mathcal{H} = \begin{bmatrix} H(0) & & & \bigcirc \\ & H\left(\frac{2\pi}{N}\right) & & \\ & & \ddots & \\ \bigcirc & & & H\left(\frac{(N-1)2\pi}{N}\right) \end{bmatrix}$$

- Since the sinewaves are orthogonal, it follows

a

$$\underline{z} = \underline{W}_N \, \underline{y} = \underline{W}_N \left(\frac{1}{N} \underline{W}_N^H \, \underline{\mathcal{H}} \, \underline{b} \right)$$

$$= \underbrace{\frac{1}{N} \underline{W}_N \underline{W}_N^H}_{\underline{I}_N} \, \underline{\mathcal{H}} \, \underline{b}$$

$$= \underline{\mathcal{H}} \, \underline{b} = \begin{bmatrix} H(0)\, b_0 \\ H\left(\frac{2\pi}{N}\right) b_1 \\ \vdots \\ H\left((N-1)\frac{2\pi}{N}\right) \end{bmatrix}$$

- and we can simply divide out the channel gains to obtain the information symbols

- Again, the FFT exploits all the structure of $\underline{W}_N$ to form the product

$$\underset{N \times 1}{\underline{z}} = \underset{N \times N}{\underline{W}_N} \; \underset{N \times 1}{\underline{y}}$$

with only $\frac{N}{2} \log_2 N$ multiplications rather than N^2 multiplications for $N = 2^v$ v, integer

- Typical numbers used in practice $N = 1024$, $N = 2048$, etc.

$\Rightarrow$ dramatic savings in computation!

See Table 8.1 on page 522 of text

- So far we have discussed the use of the Inverse DFT Matrix and the DFT Matrix in the context of OFDM

- In a more general setting, the DFT matrix is related to "sampling" the DTFT of a finite-length (real-world) causal signal at N equi-spaced frequencies between 0 and 2π

- Specifically, place the values of a DT signal of length N as the elements of an $N \times 1$ vector $\underline{x}$
- **Then:** $\underline{X}_N = \underline{W}_N \underline{x}$, where $\underline{X}_N$ is an $N \times 1$ vector containing $X_N(k) = X(\omega)\Big|_{\omega = k\frac{2\pi}{N}}$

$$k = 0, 1, \ldots, N-1$$

- Consider evaluating the DTFT of a causal DT signal of length N at the frequency $\frac{2\pi k}{N}$

$$X_N(k) = X(\omega)\Big|_{\omega = k\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}\Big|_{\omega = k\frac{2\pi}{N}}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \begin{bmatrix} 1 & e^{-jk\frac{2\pi}{N}} & e^{-jk\frac{2\pi}{N}(2)} & \cdots & e^{-jk\frac{2\pi}{N}(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$= \underline{s}^H\left(k\frac{2\pi}{N}\right)\underline{x}$$

THUS: $\underset{N\times 1}{\underline{X}_N} = \underset{N\times N}{\underline{W}_N}\ \underset{N\times 1}{\underline{x}}$

$$\underset{N\times 1}{\begin{bmatrix} X_N(0) \\ X_N(1) \\ \vdots \\ X_N(N-1) \end{bmatrix}} = \underset{N\times 1}{\begin{bmatrix} X(0) \\ X\left(\frac{2\pi}{N}\right) \\ \vdots \\ X\left((N-1)\frac{2\pi}{N}\right) \end{bmatrix}} = \underset{N\times N}{\begin{bmatrix} \underline{s}^H(0) \\ \underline{s}^H\left(\frac{2\pi}{N}\right) \\ \vdots \\ \underline{s}^H\left((N-1)\frac{2\pi}{N}\right) \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}}$$

- If $x[n]$ is of length $L < N$, then simply pad $N-L$ zeroes at end to form $N\times 1$ signal vector $\underline{x}$ $\Rightarrow$ "zero padding"