- DFT of finite length sinusoid
- $x[n] = e^{j\omega_0 n}, \quad n = 0, 1, \ldots, L-1$

  $= 0 \quad \text{otherwise}$

  $= e^{j\omega_0 n} \{u[n] - u[n-L]\}$
- First, compute DTFT of the rectangular window: $w[n] = u[n] - u[n-L]$
- then use modulation property of DTFT:

$$X(\omega) = W(\omega - \omega_0)$$

$$X_N(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

$$k = 0, 1, \ldots, N-1$$

• I. DTFT of $w[n] = u[n] - u[n-L]$

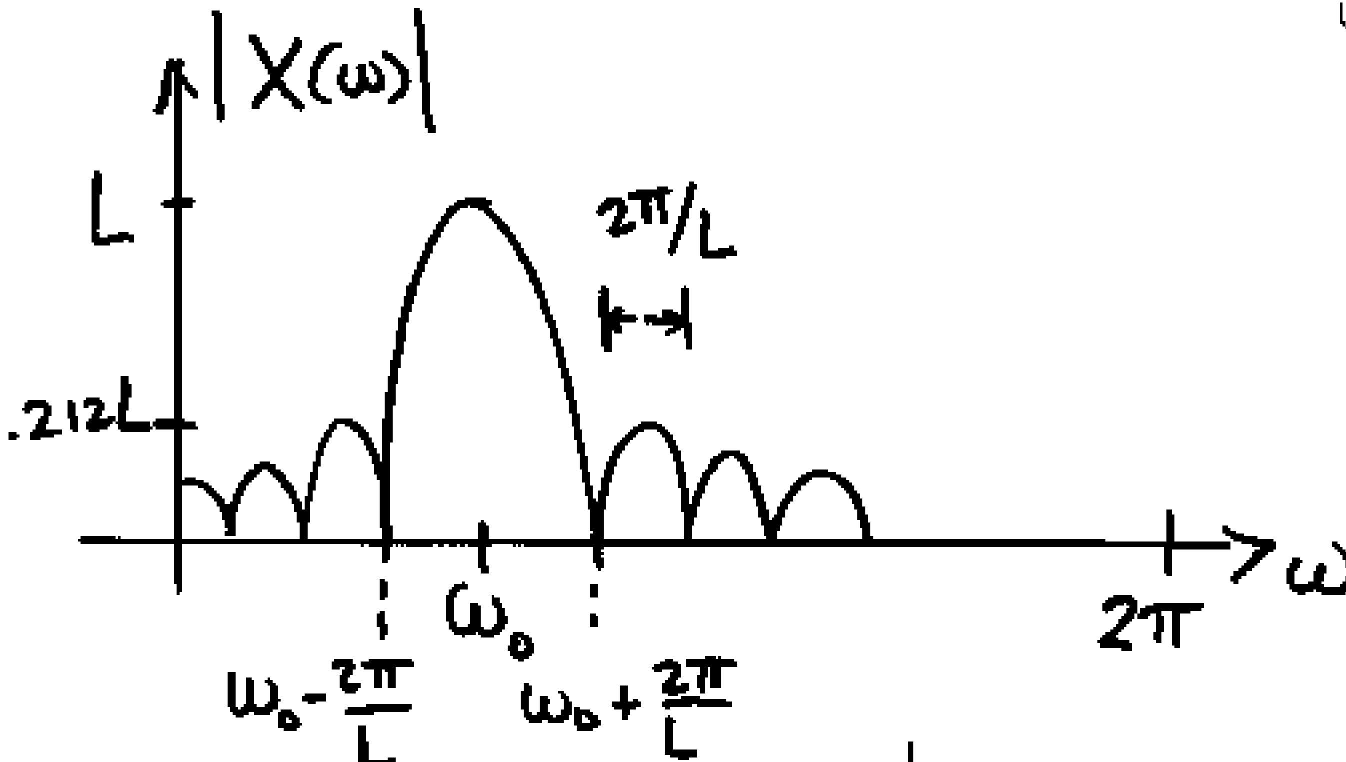
$$W(\omega) = \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\frac{L}{2}\omega}}{e^{-j\frac{\omega}{2}}} \cdot \frac{\left(e^{j\frac{L}{2}\omega} - e^{-j\frac{L}{2}\omega}\right)}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \cdot \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$W(\omega) = e^{-j\frac{(L-1)}{2}\omega} \frac{\sin\left(\frac{L}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

- thus: DTFT of finite length sinewave:

$$X(\omega) = e^{-j\frac{(L-1)}{2}(\omega-\omega_0)} \frac{\sin\left(\frac{L}{2}(\omega-\omega_0)\right)}{\sin\left(\frac{1}{2}(\omega-\omega_0)\right)}$$



• Finally: $X_N(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$

$k = 0, 1, \ldots, N-1$

• Consider three cases:

1.) $\omega_0 = \dfrac{2\pi k_0}{L}$ and $N = L$

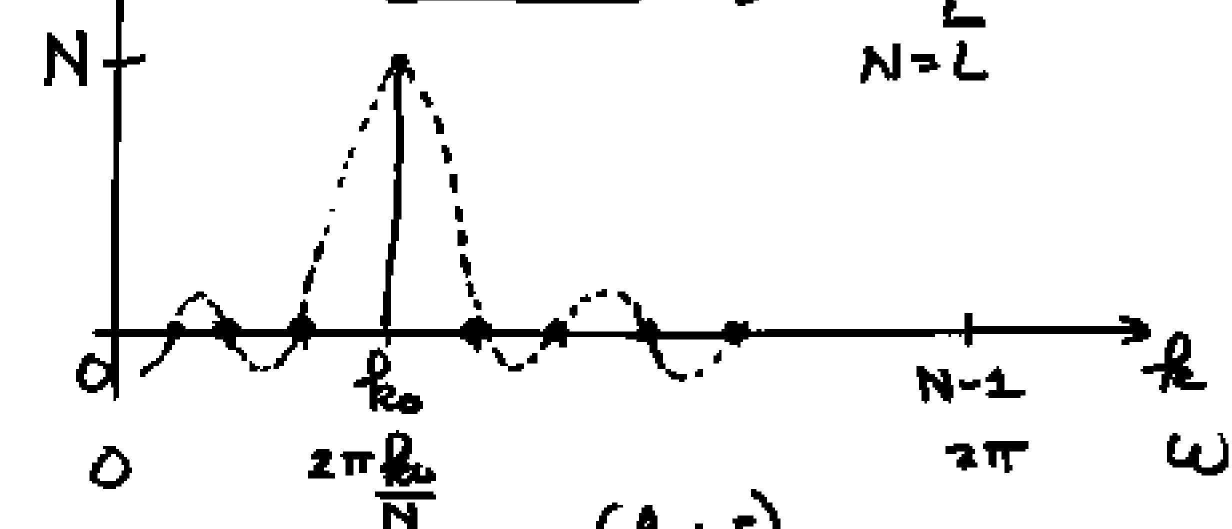
3.) $\omega_0 = \dfrac{2\pi k_0}{L}$ and $N = 2L$

2.) $\omega_0 = \dfrac{2\pi(k_0 + .5)}{L}$ and $N = L$

• See sineDFTeg1.m
sineDFTeg2.m
sineDFTeg3.m
} at course web site

Case I. $\omega_0 = 2\pi \frac{k_0}{L}$

$N = L$

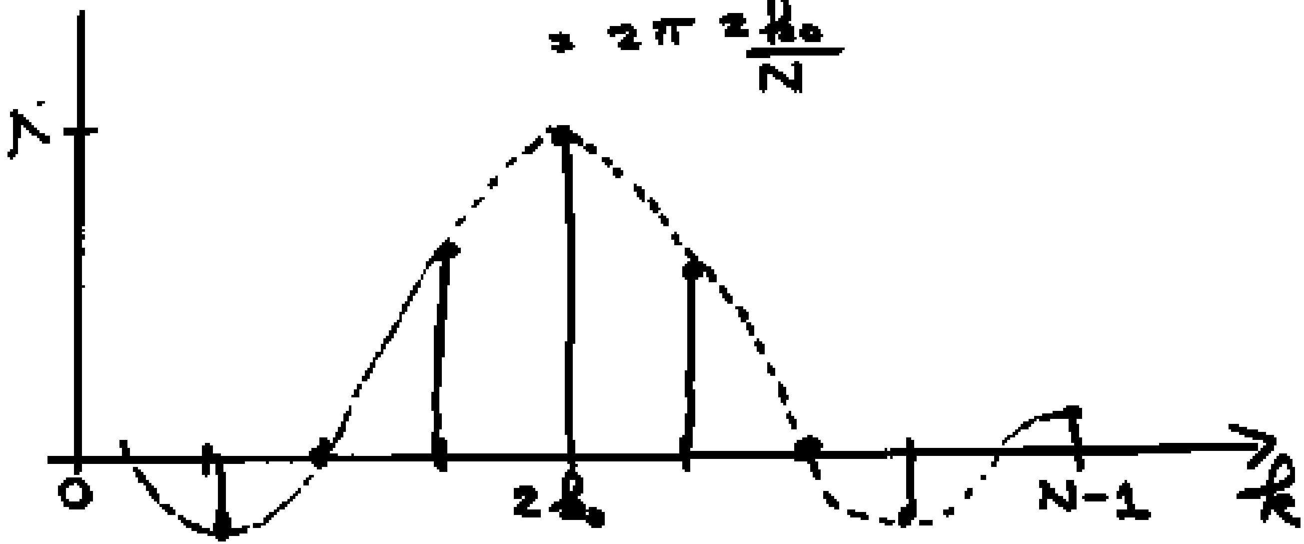


Case 2. $\omega_0 = 2\pi \frac{(k_0 + .5)}{L}$, $N = L$

$\Rightarrow$ not as lucky!

Case 3. $\omega_0 = 2\pi \frac{k_0}{L}$, $N = 2L$

$= 2\pi \frac{2k_0}{N}$



- as you increase N over L, get a better and better "picture" of DTFT $X(\omega)$ which has both mainlobe and sidelobes $\Rightarrow$ only get Dirac Delta function with $L \to \infty$ infinite length sinewave

Basic DFT Result:

$$\sum_{\ell=-\infty}^{\infty} x[n-\ell N]\, w_R(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(\omega)\Big|_{\omega = k\frac{2\pi}{N}} \triangleq X_N(k)$$

$$w_R(n) = u[n] - u[n-N] \qquad k = 0, 1, \ldots, N-1$$

where: $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

If $N > L$ = "length" of signal, then:

$$X(\omega) = \sum_{k=0}^{N-1} \underbrace{X_N(k)}_{X\left(\frac{2\pi k}{N}\right)} \frac{\sin\left(\frac{N}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)}{N \sin\left(\frac{1}{2}\left(\omega - k\frac{2\pi}{N}\right)\right)} e^{-j\frac{(N-1)}{2}\left(\omega - k\frac{2\pi}{N}\right)}$$

and no time-domain aliasing

$$e^{j\left(\frac{2\pi}{N}k_0\right)n}\{u[n]-u[n-N]\} \xleftrightarrow[N]{DFT} N\delta[k-k_0]$$

$$k_0 \in \{0, 1, \ldots, N-1\}$$

$$\cos\left(\frac{2\pi k_0}{N} n\right)\{u[n]-u[n-N]\} \xleftrightarrow[N]{DFT} \frac{N}{2}\delta[k-k_0] + \frac{N}{2}\delta[k-(N-k_0)]$$

$$\boxed{\frac{1}{2}e^{-j\frac{2\pi k_0}{N}n} \overset{\text{''}}{=} \frac{1}{2}e^{j\frac{2\pi(N-k_0)}{N}n}}$$

$$\cos\left(2\pi\frac{k_0}{N}n\right) = \frac{1}{2}e^{j2\pi\frac{k_0}{N}n} + \frac{1}{2}e^{-j2\pi\frac{k_0}{N}n}$$