CHAPTER

6

Sampling and Reconstruction of Signals

In Chapter 1 we treated the sampling of continuous-time signals and demonstrated that if the signals are bandlimited, it is possible to reconstruct the original signal from the samples, provided that the sampling rate is at least twice the highest frequency contained in the signal. We also briefly described the subsequent operations of quantization and coding that are necessary to convert an analog signal to a digital signal appropriate for digital processing.

In this chapter we consider time-domain sampling, analog-to-digital (A/D) conversion (quantization and coding), and digital-to-analog (D/A) conversion (signal reconstruction) in greater depth. We also consider the sampling of signals that are characterized as bandpass signals. The final topic deals with the use of oversampling and sigma-delta modulation in the design of high precision A/D converters.

6.1 Ideal Sampling and Reconstruction of Continuous-Time Signals

To process a continuous-time signal using digital signal processing techniques, it is necessary to convert the signal into a sequence of numbers. As was discussed in Section 1.4, this is usually done by sampling the analog signal, say $x_a(t)$, periodically every T seconds to produce a discrete-time signal $x(n)$ given by

$$x(n) = x_a(nT), \qquad -\infty < n < \infty \tag{6.1.1}$$

The relationship (6.1.1) describes the sampling process in the time domain. As discussed in Chapter 1, the sampling frequency $F_s = 1/T$ must be selected large enough such that the sampling does not cause any loss of spectral information (no aliasing). Indeed, if the spectrum of the analog signal can be recovered from the spectrum of the discrete-time signal, there is no loss of information. Consequently, we investigate the sampling process by finding the relationship between the spectra of signals $x_a(t)$ and $x(n)$.

If $x_a(t)$ is an aperiodic signal with finite energy, its (voltage) spectrum is given by the Fourier transform relation

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\, dt \qquad (6.1.2)$$

whereas the signal $x_a(t)$ can be recovered from its spectrum by the inverse Fourier transform

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\, dF \qquad (6.1.3)$$

Note that utilization of all frequency components in the infinite frequency range $-\infty < F < \infty$ is necessary to recover the signal $x_a(t)$ if the signal $x_a(t)$ is not bandlimited.

The spectrum of a discrete-time signal $x(n)$, obtained by sampling $x_a(t)$, is given by the Fourier transform relation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad (6.1.4)$$

or, equivalently,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \qquad (6.1.5)$$

The sequence $x(n)$ can be recovered from its spectrum $X(\omega)$ or $X(f)$ by the inverse transform

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\, d\omega \\ &= \int_{-1/2}^{1/2} X(f)e^{j2\pi fn}\, df \end{aligned} \qquad (6.1.6)$$

In order to determine the relationship between the spectra of the discrete-time signal and the analog signal, we note that periodic sampling imposes a relationship between the independent variables t and n in the signals $x_a(t)$ and $x(n)$, respectively. That is,

$$t = nT = \frac{n}{F_s} \qquad (6.1.7)$$

This relationship in the time domain implies a corresponding relationship between the frequency variables F and f in $X_a(F)$ and $X(f)$, respectively.

Indeed, substitution of (6.1.7) into (6.1.3) yields

$$x(n) \equiv x_a(nT) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF \tag{6.1.8}$$

If we compare (6.1.6) with (6.1.8), we conclude that

$$\int_{-1/2}^{1/2} X(f) e^{j2\pi fn}\, df = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF \tag{6.1.9}$$

From the development in Chapter 1 we know that periodic sampling imposes a relationship between the frequency variables F and f of the corresponding analog and discrete-time signals, respectively. That is,

$$f = \frac{F}{F_s} \tag{6.1.10}$$

With the aid of (6.1.10), we can make a simple change in variable in (6.1.9), and obtain the result

$$\frac{1}{F_s} \int_{-F_s/2}^{F_s/2} X(F) e^{j2\pi nF/F_s}\, dF = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF \tag{6.1.11}$$

We now turn our attention to the integral on the right-hand side of (6.1.11). The integration range of this integral can be divided into an infinite number of intervals of width F_s. Thus the integral over the infinite range can be expressed as a sum of integrals, that is,

$$\int_{-\infty}^{\infty} X_a(F) e^{j2\pi nF/F_s}\, dF = \sum_{k=-\infty}^{\infty} \int_{(k-1/2)F_s}^{(k+1/2)F_s} X_a(F) e^{j2\pi nF/F_s}\, dF \tag{6.1.12}$$

We observe that $X_a(F)$ in the frequency interval $(k - \frac{1}{2})F_s$ to $(k + \frac{1}{2})F_s$ is identical to $X_a(F - kF_s)$ in the interval $-F_s/2$ to $F_s/2$. Consequently,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \int_{(k-1/2)F_s}^{(k+1/2)F_s} X_a(F) e^{j2\pi nF/F_s}\, dF &= \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_a(F - kF_s) e^{j2\pi nF/F_s}\, dF \\ &= \int_{-F_s/2}^{F_s/2} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_s) \right] e^{j2\pi nF/F_s}\, dF \end{aligned} \tag{6.1.13}$$

where we have used the periodicity of the complex exponential, namely,

$$e^{j2\pi n(F+kF_s)/F_s} = e^{j2\pi nF/F_s}$$

Comparing (6.1.13), (6.1.12), and (6.1.11), we conclude that

$$X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \qquad (6.1.14)$$

or, equivalently,

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f - k)F_s] \qquad (6.1.15)$$

This is the desired relationship between the spectrum $X(F)$ or $X(f)$ of the discrete-time signal and the spectrum $X_a(F)$ of the analog signal. The right-hand side of (6.1.14) or (6.1.15) consists of a periodic repetition of the scaled spectrum $F_s X_a(F)$ with period F_s. This periodicity is necessary because the spectrum $X(f)$ of the discrete-time signal is periodic with period $f_p = 1$ or $F_p = F_s$.

For example, suppose that the spectrum of a band-limited analog signal is as shown in Fig. 6.1.1(a). The spectrum is zero for $|F| \geq B$. Now, if the sampling frequency F_s is selected to be greater than $2B$, the spectrum $X(F_s)$ of the discrete-time signal will appear as shown in Fig. 6.1.1(b). Thus, if the sampling frequency F_s is selected such that $F_s \geq 2B$, where $2B$ is the Nyquist rate, then

$$X(F) = F_s X_a(F), \qquad |F| \leq F_s/2 \qquad (6.1.16)$$

In this case there is no aliasing and therefore the spectrum of the discrete-time signal is identical (within the scale factor F_s) to the spectrum of the analog signal, within the fundamental frequency range $|F| \leq F_s/2$ or $|f| \leq \frac{1}{2}$.

On the other hand, if the sampling frequency F_s is selected such that $F_s < 2B$, the periodic continuation of $X_a(F)$ results in spectral overlap, as illustrated in Fig. 6.1.1(c) and (d). Thus the spectrum $X(F)$ of the discrete-time signal contains aliased frequency components of the analog signal spectrum $X_a(F)$. The end result is that the aliasing which occurs prevents us from recovering the original signal $x_a(t)$ from the samples.

Given the discrete-time signal $x(n)$ with the spectrum $X(F)$, as illustrated in Fig. 6.1.1(b), with no aliasing, it is now possible to reconstruct the original analog signal from the samples $x(n)$. Since in the absence of aliasing

$$X_a(F) = \begin{cases} \dfrac{1}{F_s} X(F), & |F| \leq F_s/2 \\ 0, & |F| > F_s/2 \end{cases} \qquad (6.1.17)$$

and by the Fourier transform relationship (6.1.5),

$$X(F) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi Fn/F_s} \qquad (6.1.18)$$

the inverse Fourier transform of $X_a(F)$ is

$$x_a(t) = \int_{-F_s/2}^{F_s/2} X_a(F) e^{j2\pi Ft}\, dF \qquad (6.1.19)$$

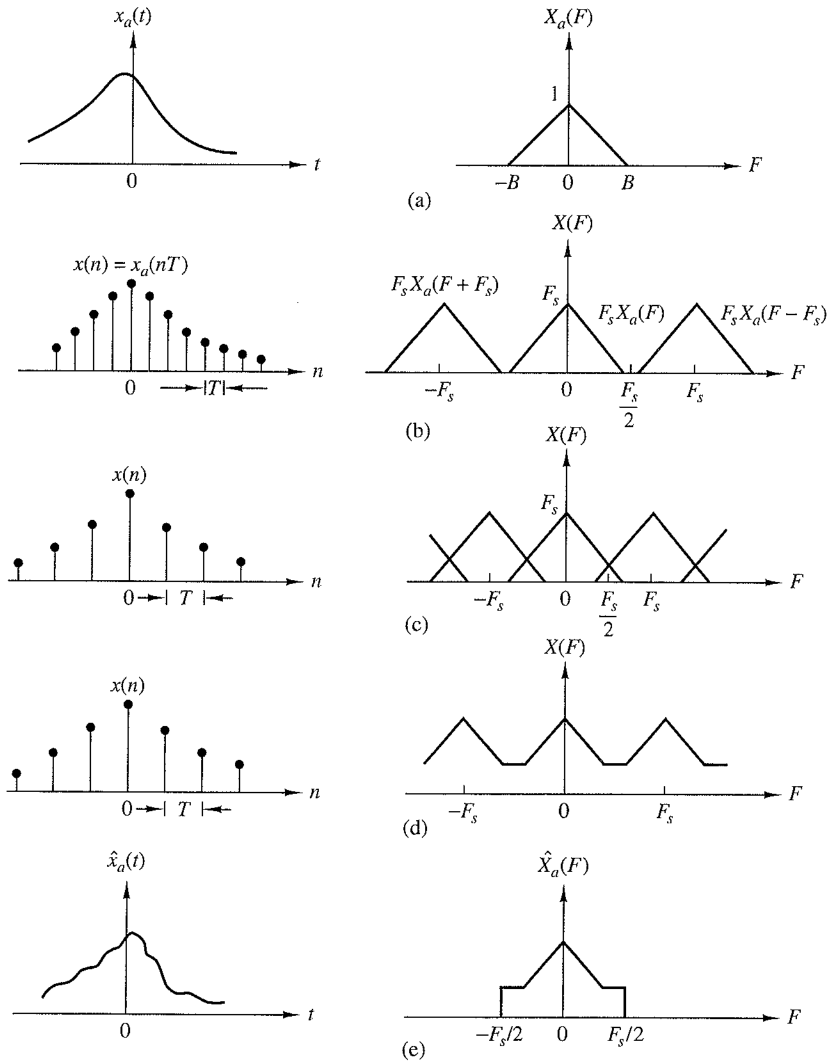

Figure 6.1.1 Sampling of an analog bandlimited signal and aliasing of spectral components.

Let us assume that $F_s \geq 2B$. With the substitution of (6.1.17) into (6.1.19), we have

$$
\begin{aligned}
x_a(t) &= \frac{1}{F_s}\int_{-F_s/2}^{F_s/2}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi Fn/F_s}\right]e^{j2\pi Ft}\,dF \\
&= \frac{1}{F_s}\sum_{n=-\infty}^{\infty} x(n)\int_{-F_s/2}^{F_s/2} e^{j2\pi F(t-n/F_s)}\,dF \qquad (6.1.20)\\
&= \sum_{n=-\infty}^{\infty} x_a(nT)\frac{\sin(\pi/T)(t-nT)}{(\pi/T)(t-nT)}
\end{aligned}
$$

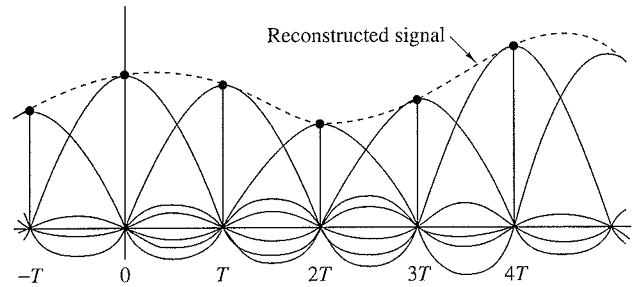

Figure 6.1.2
Reconstruction of a continuous-time signal using ideal interpolation.

where $x(n) = x_a(nT)$ and where $T = 1/F_s$ is the sampling interval. This is the reconstruction formula given by (1.4.24) in our discussion of the sampling theorem.

The reconstruction formula in (6.1.20) involves the function

$$g(t) = \frac{\sin(\pi/T)t}{(\pi/T)t} \tag{6.1.21}$$

appropriately shifted by nT, $n = 0, \pm 1, \pm 2, \ldots$, and multiplied or weighted by the corresponding samples $x_a(nT)$ of the signal. We call (6.1.20) an interpolation formula for reconstructing $x_a(t)$ from its samples, and $g(t)$, given in (6.1.21), is the interpolation function. We note that at $t = kT$, the interpolation function $g(t - nT)$ is zero except at $k = n$. Consequently, $x_a(t)$ evaluated at $t = kT$ is simply the sample $x_a(kT)$. At all other times the weighted sums of the time-shifted versions of the interpolation function combine to yield exactly $x_a(t)$. This combination is illustrated in Fig. 6.1.2.

The formula in (6.1.20) for reconstructing the analog signal $x_a(t)$ from its samples is called the *ideal interpolation formula.* It forms the basis for the *sampling theorem,* which can be stated as follows.

Sampling Theorem. A bandlimited continuous-time signal, with highest frequency (bandwidth) B hertz, can be uniquely recovered from its samples provided that the sampling rate $F_s \geq 2B$ samples per second.

According to the sampling theorem and the reconstruction formula in (6.1.20), the recovery of $x_a(t)$ from its samples $x(n)$ requires an infinite number of samples. However, in practice we use a finite number of samples of the signal and deal with finite-duration signals. As a consequence, we are concerned only with reconstructing a finite-duration signal from a finite number of samples.

When aliasing occurs due to too low a sampling rate, the effect can be described by a multiple folding of the frequency axis of the frequency variable F for the analog signal. Figure 6.1.3(a) shows the spectrum $X_a(F)$ of an analog signal. According to (6.1.14), sampling of the signal with a sampling frequency F_s results in a periodic repetition of $X_a(F)$ with period F_s. If $F_s < 2B$, the shifted replicas of $X_a(F)$ overlap. The overlap that occurs within the fundamental frequency range $-F_s/2 \leq F \leq F_s/2$ is illustrated in Fig. 6.1.3(b). The corresponding spectrum of the discrete-time signal within the fundamental frequency range is obtained by adding all the shifted portions within the range $|f| \leq \frac{1}{2}$, to yield the spectrum shown in Fig. 6.1.3(c).

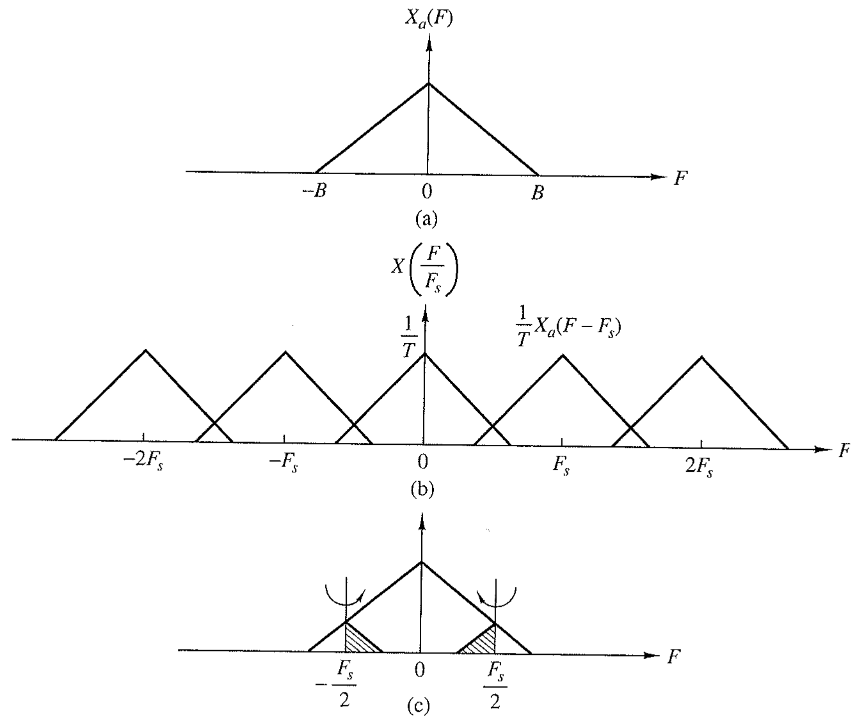

Figure 6.1.3 Illustration of aliasing around the folding frequency.

A careful inspection of Fig. 6.1.3(a) and (b) reveals that the aliased spectrum in Fig. 6.1.3(c) can be obtained by folding the original spectrum like an accordion with pleats at every odd multiple of $F_s/2$. Consequently, the frequency $F_s/2$ is called the *folding frequency*, as indicated in Chapter 1. Clearly, then, periodic sampling automatically forces a folding of the frequency axis of an analog signal at odd multiples of $F_s/2$, and this results in the relationship $F = fF_s$ between the frequencies for continuous-time signals and discrete-time signals. Due to the folding of the frequency axis, the relationship $F = fF_s$ is not truly linear, but piecewise linear, to accommodate for the aliasing effect. This relationship is illustrated in Fig. 6.1.4.

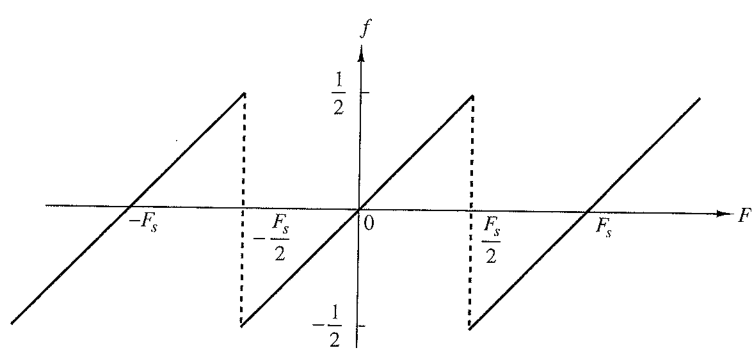

Figure 6.1.4 Relationship between frequency variables F and f.

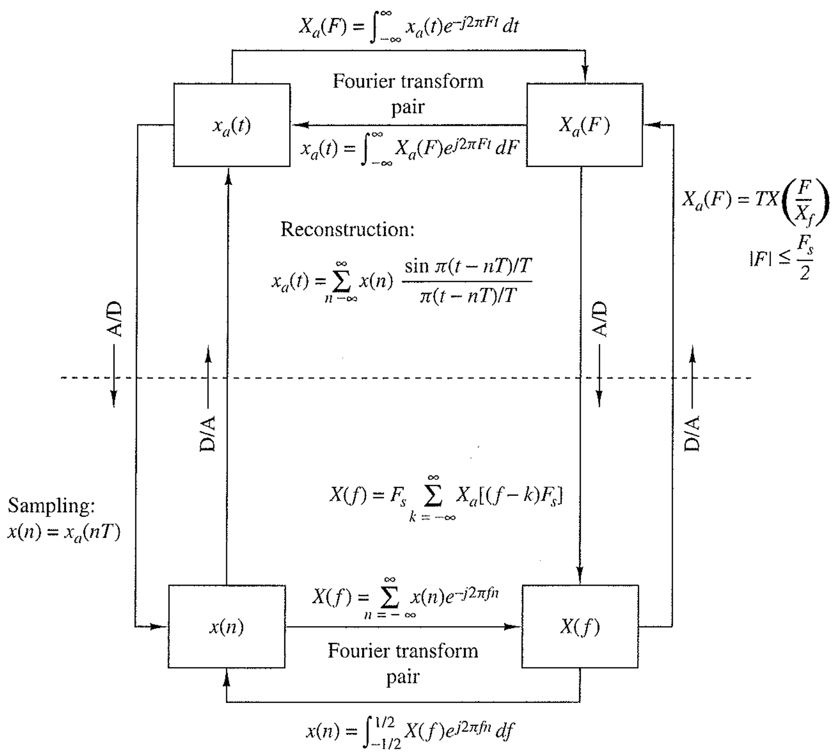

Figure 6.1.5 Time-domain and frequency-domain relationships for sampled signals.

If the analog signal is bandlimited to $B \le F_s/2$, the relationship between f and F is linear and one-to-one. In other words, there is no aliasing. In practice, prefiltering with an antialiasing filter is usually employed prior to sampling. This ensures that frequency components of the signal above $F \ge B$ are sufficiently attenuated so that, if aliased, they cause negligible distortion on the desired signal.

The relationships among the time-domain and frequency-domain functions $x_a(t)$, $x(n)$, $X_a(F)$, and $X(f)$ are summarized in Fig. 6.1.5. The relationships for recovering the continuous-time functions, $x_a(t)$ and $X_a(F)$, from the discrete-time quantities $x(n)$ and $X(f)$, assume that the analog signal is bandlimited and that it is sampled at the Nyquist rate (or faster).

The following examples serve to illustrate the problem of the aliasing of frequency components.

EXAMPLE 6.1.1 **Aliasing in Sinusoidal Signals**

The continuous-time signal

$$x_a(t) = \cos 2\pi F_0 t = \frac{1}{2}e^{j2\pi F_0 t} + \frac{1}{2}e^{-j2\pi F_0 t}$$

has a discrete spectrum with spectral lines at $F = \pm F_0$, as shown in Fig. 6.1.6(a). The process

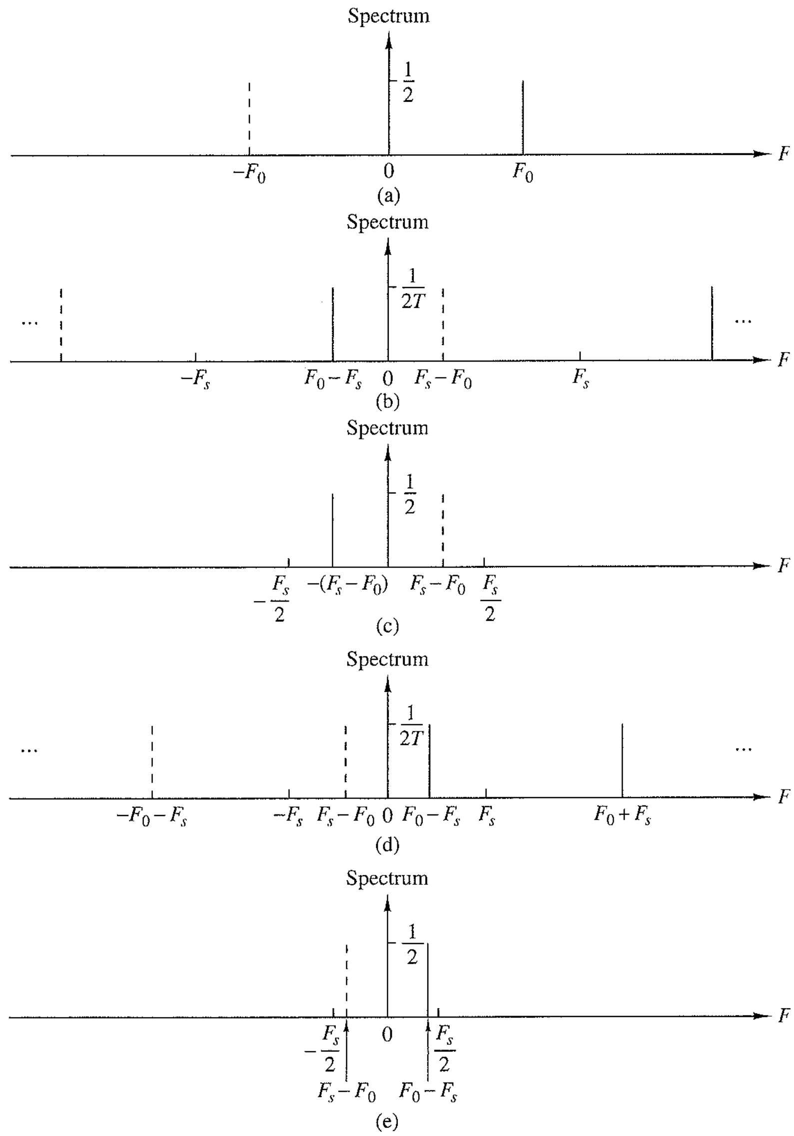

Figure 6.1.6 Aliasing of sinusoidal signals.

of sampling this signal with a sampling frequency F_s introduces replicas of the spectrum about multiples of F_s. This is illustrated in Fig. 6.1.6(b) for $F_s/2 < F_0 < F_s$.

To reconstruct the continuous-time signal, we should select the frequency components inside the fundamental frequency range $|F| \leq F_s/2$. The resulting spectrum is shown in Fig. 6.1.6(c). The reconstructed signal is

$$\hat{x}_a(t) = \cos 2\pi(F_s - F_0)t$$