5.1.2 Steady-State and Transient Response to Sinusoidal Input Signals

In the discussion in the preceding section, we determined the response of a linear time-invariant system to exponential and sinusoidal input signals applied to the system at $n = -\infty$. We usually call such signals eternal exponentials or eternal sinusoids, because they were applied at $n = -\infty$. In such a case, the response that we observe at the output of the system is the steady-state response. There is no transient response in this case.

On the other hand, if the exponential or sinusoidal signal is applied at some finite time instant, say at $n = 0$, the response of the system consists of two terms, the transient response and the steady-state response. To demonstrate this behavior, let us consider, as an example, the system described by the first-order difference equation

$$y(n) = ay(n-1) + x(n) \tag{5.1.20}$$

This system was considered in Section 2.4.2. Its response to any input $x(n)$ applied at $n = 0$ is given by (2.4.8) as

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k), \qquad n \geq 0 \tag{5.1.21}$$

where $y(-1)$ is the initial condition.

Now, let us assume that the input to the system is the complex exponential

$$x(n) = Ae^{j\omega n}, \qquad n \geq 0 \tag{5.1.22}$$

applied at $n = 0$. When we substitute (5.1.22) into (5.1.21), we obtain

$$\begin{aligned} y(n) &= a^{n+1}y(-1) + A\sum_{k=0}^{n} a^k e^{j\omega(n-k)} \\ &= a^{n+1}y(-1) + A\left[\sum_{k=0}^{n}(ae^{-j\omega})^k\right]e^{j\omega n} \\ &= a^{n+1}y(-1) + A\frac{1-a^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n}, \qquad n \geq 0 \\ &= a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} + \frac{A}{1-ae^{-j\omega}}e^{j\omega n}, \qquad n \geq 0 \end{aligned} \tag{5.1.23}$$

We recall that the system in (5.1.20) is BIBO stable if $|a| < 1$. In this case the two terms involving a^{n+1} in (5.1.23) decay toward zero as n approaches infinity. Consequently, we are left with the steady-state response

$$\begin{aligned} y_{ss}(n) = \lim_{n\to\infty} y(n) &= \frac{A}{1-ae^{-j\omega}}e^{j\omega n} \\ &= AH(\omega)e^{j\omega n} \end{aligned} \tag{5.1.24}$$

The first two terms in (5.1.23) constitute the transient response of the system, that is,

$$y_{\text{tr}}(n) = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}}e^{j\omega n}, \qquad n \geq 0 \tag{5.1.25}$$

which decay toward zero as n approaches infinity. The first term in the transient response is the zero-input response of the system and the second term is the transient produced by the exponential input signal.

In general, all linear time-invariant BIBO systems behave in a similar fashion when excited by a complex exponential, or by a sinusoid at $n = 0$ or at some other finite time instant. That is, the transient response decays toward zero as $n \to \infty$, leaving only the steady-state response that we determined in the preceding section. In many practical applications, the transient response of the system is unimportant, and therefore it is usually ignored in dealing with the response of the system to sinusoidal inputs.

5.1.3 Steady-State Response to Periodic Input Signals

Suppose that the input to a stable linear time-invariant system is a periodic signal $x(n)$ with fundamental period N. Since such a signal exists from $-\infty < n < \infty$, the total response of the system at any time instant n is simply equal to the steady-state response.

To determine the response $y(n)$ of the system, we make use of the Fourier series representation of the periodic signal, which is

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1 \tag{5.1.26}$$

where the $\{c_k\}$ are the Fourier series coefficients. Now the response of the system to the complex exponential signal

$$x_k(n) = c_k e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

is

$$y_k(n) = c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1 \tag{5.1.27}$$

where

$$H\left(\frac{2\pi k}{N}\right) = H(\omega)|_{\omega=2\pi k/N}, \qquad k = 0, 1, \ldots, N-1$$

By using the superposition principle for linear systems, we obtain the response of the system to the periodic signal $x(n)$ in (5.1.26) as

$$y(n) = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}, \qquad -\infty < n < \infty \tag{5.1.28}$$

This result implies that the response of the system to the periodic input signal $x(n)$ is also periodic with the same period N. The Fourier series coefficients for $y(n)$ are

$$d_k \equiv c_k H\left(\frac{2\pi k}{N}\right), \qquad k = 0, 1, \ldots, N-1 \tag{5.1.29}$$

Hence, the linear system can change the shape of the periodic input signal by scaling the amplitude and shifting the phase of the Fourier series components, but it does not affect the period of the periodic input signal.

5.1.4 Response to Aperiodic Input Signals

The convolution theorem, given in (4.4.49), provides the desired frequency-domain relationship for determining the output of an LTI system to an aperiodic finite-energy signal. If $\{x(n)\}$ denotes the input sequence, $\{y(n)\}$ denotes the output sequence, and $\{h(n)\}$ denotes the unit sample response of the system, then from the convolution theorem, we have

$$Y(\omega) = H(\omega)X(\omega) \tag{5.1.30}$$

where $Y(\omega)$, $X(\omega)$, and $H(\omega)$ are the corresponding Fourier transforms of $\{y(n)\}$, $\{x(n)\}$, and $\{h(n)\}$, respectively. From this relationship we observe that the spectrum of the output signal is equal to the spectrum of the input signal multiplied by the frequency response of the system.

If we express $Y(\omega)$, $H(\omega)$, and $X(\omega)$ in polar form, the magnitude and phase of the output signal can be expressed as

$$|Y(\omega)| = |H(\omega)||X(\omega)| \tag{5.1.31}$$

$$\measuredangle Y(\omega) = \measuredangle X(\omega) + \measuredangle H(\omega) \tag{5.1.32}$$

where $|H(\omega)|$ and $\measuredangle H(\omega)$ are the magnitude and phase responses of the system.

By its very nature, a finite-energy aperiodic signal contains a continuum of frequency components. The linear time-invariant system, through its frequency response function, attenuates some frequency components of the input signal and amplifies other frequency components. Thus the system acts as a *filter*to the input signal. Observation of the graph of $|H(\omega)|$ shows which frequency components are amplified and which are attenuated. On the other hand, the angle of $H(\omega)$ determines the phase shift imparted in the continuum of frequency components of the input signal as a function of frequency. If the input signal spectrum is changed by the system in an undesirable way, we say that the system has caused *magnitude and phase distortion.*

We also observe that *the output of a linear time-invariant system cannot contain frequency components that are not contained in the input signal.* It takes either a linear time-variant system or a nonlinear system to create frequency components that are not necessarily contained in the input signal.

Figure 5.1.3 illustrates the time-domain and frequency-domain relationships that can be used in the analysis of BIBO-stable LTI systems. We observe that in time-domain analysis, we deal with the convolution of the input signal with the impulse

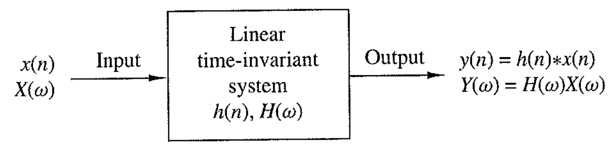

Figure 5.1.3
Time- and frequency-domain input–output relationships in LTI systems.

response of the system to obtain the output sequence of the system. On the other hand, in frequency-domain analysis, we deal with the input signal spectrum $X(\omega)$ and the frequency response $H(\omega)$ of the system, which are related through multiplication, to yield the spectrum of the signal at the output of the system.

We can use the relation in (5.1.30) to determine the spectrum $Y(\omega)$ of the output signal. Then the output sequence $\{y(n)\}$ can be determined from the inverse Fourier transform

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega n}d\omega \tag{5.1.33}$$

However, this method is seldom used. Instead, the z-transform introduced in Chapter 3 is a simpler method for solving the problem of determining the output sequence $\{y(n)\}$.

Let us return to the basic input–output relation in (5.1.30) and compute the squared magnitude of both sides. Thus we obtain

$$\begin{aligned} |Y(\omega)|^2 &= |H(\omega)|^2|X(\omega)|^2 \\ S_{yy}(\omega) &= |H(\omega)|^2 S_{xx}(\omega) \end{aligned} \tag{5.1.34}$$

where $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are the energy density spectra of the input and output signals, respectively. By integrating (5.1.34) over the frequency range $(-\pi, \pi)$, we obtain the energy of the output signal as

$$\begin{aligned} E_y &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 S_{xx}(\omega)d\omega \end{aligned} \tag{5.1.35}$$

EXAMPLE 5.1.5

A linear time-invariant system is characterized by its impulse response

$$h(n) = (\frac{1}{2})^n u(n)$$

Determine the spectrum and the energy density spectrum of the output signal when the system is excited by the signal

$$x(n) = (\frac{1}{4})^n u(n)$$

Solution. The frequency response function of the system

$$H(\omega) = \sum_{n=0}^{\infty}(\frac{1}{2})^n e^{-j\omega n}$$
$$= \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Similarly, the input sequence $\{x(n)\}$ has a Fourier transform

$$X(\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}$$

Hence the spectrum of the signal at the output of the system is

$$Y(\omega) = H(\omega)X(\omega)$$
$$= \frac{1}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

The corresponding energy density spectrum is

$$S_{yy}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2$$
$$= \frac{1}{(\frac{5}{4} - \cos\omega)(\frac{17}{16} - \frac{1}{2}\cos\omega)}$$

5.2 Frequency Response of LTI Systems

In this section we focus on determining the frequency response of LTI systems that have rational system functions. Recall that this class of LTI systems is described in the time domain by constant-coefficient difference equations.

5.2.1 Frequency Response of a System with a Rational System Function

From the discussion in Section 4.2.6 we know that if the system function $H(z)$ converges on the unit circle, we can obtain the frequency response of the system by evaluating $H(z)$ on the unit circle. Thus

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \tag{5.2.1}$$

In the case where $H(z)$ is a rational function of the form $H(z) = B(z)/A(z)$, we have

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} \tag{5.2.2}$$

$$= b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})} \tag{5.2.3}$$

where the $\{a_k\}$ and $\{b_k\}$ are real, but $\{z_k\}$ and $\{p_k\}$ may be complex valued.

It is sometimes desirable to express the magnitude squared of $H(\omega)$ in terms of $H(z)$. First, we note that

$$|H(\omega)|^2 = H(\omega)H^*(\omega)$$

For the rational system function given by (5.2.3), we have

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k^* e^{j\omega})}{\prod_{k=1}^{N} (1 - p_k^* e^{j\omega})} \tag{5.2.4}$$

It follows that $H^*(\omega)$ is obtained by evaluating $H^*(1/z^*)$ on the unit circle, where for a rational system function,

$$H^*(1/z^*) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k^* z)}{\prod_{k=1}^{N} (1 - p_k^* z)} \tag{5.2.5}$$

However, when $\{h(n)\}$ is real or, equivalently, the coefficients $\{a_k\}$ and $\{b_k\}$ are real, complex-valued poles and zeros occur in complex-conjugate pairs. In this case, $H^*(1/z^*) = H(z^{-1})$. Consequently, $H^*(\omega) = H(-\omega)$, and

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})|_{z=e^{j\omega}} \tag{5.2.6}$$

According to the correlation theorem for the z-transform (see Table 3.2), the function $H(z)H(z^{-1})$ is the z-transform of the autocorrelation sequence $\{r_{hh}(m)\}$

of the unit sample response $\{h(n)\}$. Then it follows from the Wiener–Khintchine theorem that $|H(\omega)|^2$ is the Fourier transform of $\{r_{hh}(m)\}$.

Similarly, if $H(z) = B(z)/A(z)$, the transforms $D(z) = B(z)B(z^{-1})$ and $C(z) = A(z)A(z^{-1})$ are the z-transforms of the autocorrelation sequences $\{c_l\}$ and $\{d_l\}$, where

$$c_l = \sum_{k=0}^{N-|l|} a_k a_{k+l}, \qquad -N \le l \le N \tag{5.2.7}$$

$$d_l = \sum_{k=0}^{M-|l|} b_k b_{k+l}, \qquad -M \le l \le M \tag{5.2.8}$$

Since the system parameters $\{a_k\}$ and $\{b_k\}$ are real valued, it follows that $c_l = c_{-l}$ and $d_l = d_{-l}$. By using this symmetry property, $|H(\omega)|^2$ may be expressed as

$$|H(\omega)|^2 = \frac{d_0 + 2\sum_{k=1}^{M} d_k \cos k\omega}{c_0 + 2\sum_{k=1}^{N} c_k \cos k\omega} \tag{5.2.9}$$

Finally, we note that $\cos k\omega$ can be expressed as a polynomial function of $\cos\omega$. That is,

$$\cos k\omega = \sum_{m=0}^{k} \beta_m (\cos\omega)^m \tag{5.2.10}$$

where $\{\beta_m\}$ are the coefficients in the expansion. Consequently, the numerator and denominator of $|H(\omega)|^2$ can be viewed as polynomial functions of $\cos\omega$. The following example illustrates the foregoing relationships.

EXAMPLE 5.2.1

Determine $|H(\omega)|^2$ for the system

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + x(n) + x(n-1)$$

Solution. The system function is

$$H(z) = \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}$$

and its ROC is $|z| > 0.5$. Hence $H(\omega)$ exists. Now

$$\begin{aligned} H(z)H(z^{-1}) &= \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}} \cdot \frac{1+z}{1+0.1z-0.2z^2} \\ &= \frac{2+z+z^{-1}}{1.05+0.08(z+z^{-1})-0.2(z^{-2}+z^{-2})} \end{aligned}$$

By evaluating $H(z)H(z^{-1})$ on the unit circle, we obtain

$$|H(\omega)|^2 = \frac{2 + 2\cos\omega}{1.05 + 0.16\cos\omega - 0.4\cos 2\omega}$$

However, $\cos 2\omega = 2\cos^2\omega - 1$. Consequently, $|H(\omega)|^2$ may be expressed as

$$|H(\omega)|^2 = \frac{2(1 + \cos\omega)}{1.45 + 0.16\cos\omega - 0.8\cos^2\omega}$$

We note that given $H(z)$, it is straightforward to determine $H(z^{-1})$ and then $|H(\omega)|^2$. However, the inverse problem of determining $H(z)$, given $|H(\omega)|^2$ or the corresponding impulse response $\{h(n)\}$, is not straightforward. Since $|H(\omega)|^2$ does not contain the phase information in $H(\omega)$, it is not possible to uniquely determine $H(z)$.

To elaborate on the point, let us assume that the N poles and M zeros of $H(z)$ are $\{p_k\}$ and $\{z_k\}$, respectively. The corresponding poles and zeros of $H(z^{-1})$ are $\{1/p_k\}$ and $\{1/z_k\}$, respectively. Given $|H(\omega)|^2$ or, equivalently, $H(z)H(z^{-1})$, we can determine different system functions $H(z)$ by assigning to $H(z)$, a pole p_k or its reciprocal $1/p_k$, and a zero z_k or its reciprocal $1/z_k$. For example, if $N = 2$ and $M = 1$, the poles and zeros of $H(z)H(z^{-1})$ are $\{p_1, p_2, 1/p_1, 1/p_2\}$ and $\{z_1, 1/z_1\}$. If p_1 and p_2 are real, the possible poles for $H(z)$ are $\{p_1, p_2\}$, $\{1/p_1, 1/p_2\}$, $\{p_1, 1/p_2\}$, and $\{p_2, 1/p_1\}$ and the possible zeros are $\{z_1\}$ or $\{1/z_1\}$. Therefore, there are eight possible choices of system functions, all of which result in the same $|H(\omega)|^2$. Even if we restrict the poles of $H(z)$ to be inside the unit circle, there are still two different choices for $H(z)$, depending on whether we pick the zero $\{z_1\}$ or $\{1/z_1\}$. Therefore, we cannot determine $H(z)$ uniquely given only the magnitude response $|H(\omega)|$.

5.2.2 Computation of the Frequency Response Function

In evaluating the magnitude response and the phase response as functions of frequency, it is convenient to express $H(\omega)$ in terms of its poles and zeros. Hence we write $H(\omega)$ in factored form as

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega k})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega k})} \tag{5.2.11}$$

or, equivalently, as

$$H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)} \tag{5.2.12}$$

Let us express the complex-valued factors in (5.2.12) in polar form as

$$e^{j\omega} - z_k = V_k(\omega)e^{j\Theta_k(\omega)} \tag{5.2.13}$$

and

$$e^{j\omega} - p_k = U_k(\omega)e^{j\Phi_k(\omega)} \tag{5.2.14}$$

where

$$V_k(\omega) \equiv |e^{j\omega} - z_k|, \quad \Theta_k(\omega) \equiv \measuredangle(e^{j\omega} - z_k) \tag{5.2.15}$$

and

$$U_k(\omega) \equiv |e^{j\omega} - p_k|, \quad \Phi_k(\omega) = \measuredangle(e^{j\omega} - p_k) \tag{5.2.16}$$

The magnitude of $H(\omega)$ is equal to the product of magnitudes of all terms in (5.2.12). Thus, using (5.2.13) through (5.2.16), we obtain

$$|H(\omega)| = |b_0| \frac{V_1(\omega) \cdots V_M(\omega)}{U_1(\omega)U_2(\omega) \cdots U_N(\omega)} \tag{5.2.17}$$

since the magnitude of $e^{j\omega(N-M)}$ is 1.

The phase of $H(\omega)$ is the sum of the phases of the numerator factors, minus the phases of the denominator factors. Thus, by combining (5.2.13) through (5.2.16), we have

$$\begin{aligned} \measuredangle H(\omega) = \measuredangle b_0 + \omega(N-M) + \Theta_1(\omega) + \Theta_2(\omega) + \cdots + \Theta_M(\omega) \\ - [\Phi_1(\omega) + \Phi_2(\omega) + \cdots + \Phi_N(\omega)] \end{aligned} \tag{5.2.18}$$

The phase of the gain term b_0 is zero or π, depending on whether b_0 is positive or negative. Clearly, if we know the zeros and the poles of the system function $H(z)$, we can evaluate the frequency response from (5.2.17) and (5.2.18).

There is a geometric interpretation of the quantities appearing in (5.2.17) and (5.2.18). Let us consider a pole p_k and a zero z_k located at points A and B of the z-plane, as shown in Fig. 5.2.1(a). Assume that we wish to compute $H(\omega)$ at a specific value of frequency ω. The given value of ω determines the angle of $e^{j\omega}$ with the positive real axis. The tip of the vector $e^{j\omega}$ specifies a point L on the unit circle. The evaluation of the Fourier transform for the given value of ω is equivalent to evaluating the z-transform at the point L of the complex plane. Let us draw the vectors **AL** and **BL** from the pole and zero locations to the point L, at which we wish to compute the Fourier transform. From Fig. 5.2.1(a) it follows that

$$\mathbf{CL} = \mathbf{CA} + \mathbf{AL}$$

and

$$\mathbf{CL} = \mathbf{CB} + \mathbf{BL}$$

However, $\mathbf{CL} = e^{j\omega}$, $\mathbf{CA} = p_k$ and $\mathbf{CB} = z_k$. Thus

$$\mathbf{AL} = e^{j\omega} - p_k \tag{5.2.19}$$

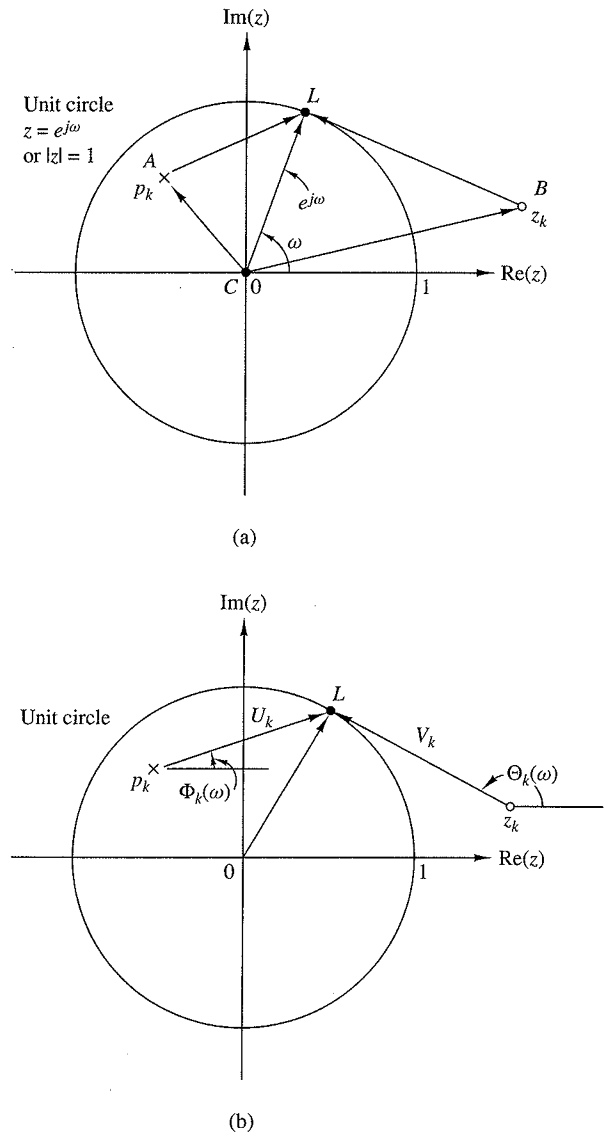

Figure 5.2.1
Geometric interpretation of the contribution of a pole and a zero to the Fourier transform (1) magnitude: the factor V_k/U_k, (2) phase: the factor $\Theta_k - \Phi_k$.

and

$$\mathbf{BL} = e^{j\omega} - z_k \tag{5.2.20}$$

By combining these relations with (5.2.13) and (5.2.14), we obtain

$$\mathbf{AL} = e^{j\omega} - p_k = U_k(\omega)e^{j\Phi_k(\omega)} \tag{5.2.21}$$

$$\mathbf{BL} = e^{j\omega} - z_k = V_k(\omega)e^{j\Theta_k(\omega)} \tag{5.2.22}$$

Thus $U_k(\omega)$ is the length of **AL**, that is, the distance of the pole p_k from the point L corresponding to $e^{j\omega}$, whereas $V_k(\omega)$ is the distance of the zero z_k from the same lar form as

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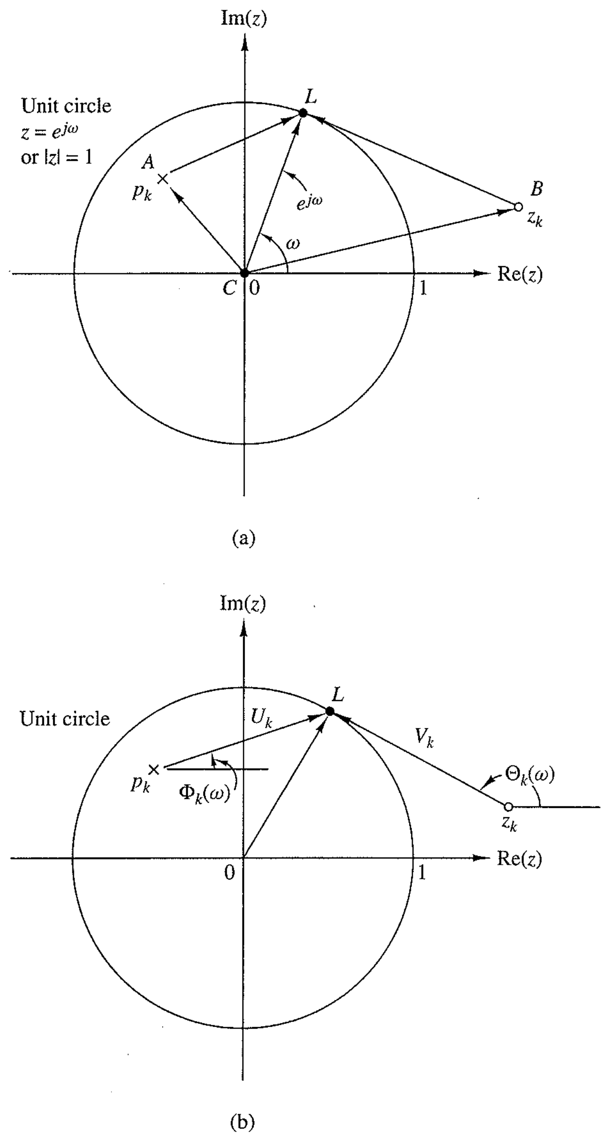

Figure 5.2.1
Geometric interpretation of the contribution of a pole and a zero to the Fourier transform (1) magnitude: the factor V_k/U_k, (2) phase: the factor $\Theta_k - \Phi_k$.

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$$\mathbf{BL} = e^{j\omega} - z_k \tag{5.2.20}$$

By combining these relations with (5.2.13) and (5.2.14), we obtain

$$\mathbf{AL} = e^{j\omega} - p_k = U_k(\omega)e^{j\Phi_k(\omega)} \tag{5.2.21}$$

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Thus $U_k(\omega)$ is the length of **AL**, that is, the distance of the pole p_k from the point L corresponding to $e^{j\omega}$, whereas $V_k(\omega)$ is the distance of the zero z_k from the same

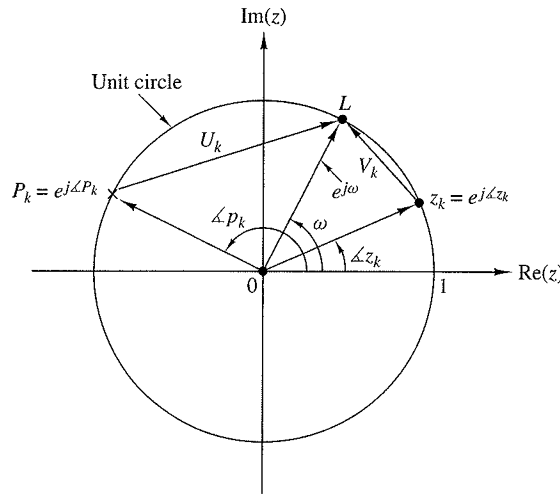

Figure 5.2.2
A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

point L. The phases $\Phi_k(\omega)$ and $\Theta_k(\omega)$ are the angles of the vectors **AL** and **BL** with the positive real axis, respectively. These geometric interpretations are shown in Fig. 5.2.1(b).

Geometric interpretations are very useful in understanding how the location of poles and zeros affects the magnitude and phase of the Fourier transform. Suppose that a zero, say z_k, and a pole, say p_k, are on the unit circle as shown in Fig. 5.2.2. We note that at $\omega = \measuredangle z_k$, $V_k(\omega)$ and consequently $|H(\omega)|$ become zero. Similarly, at $\omega = \measuredangle p_k$ the length $U_k(\omega)$ becomes zero and hence $|H(\omega)|$ becomes infinite. Clearly, the evaluation of phase in these cases has no meaning.

From this discussion we can easily see that the presence of a zero close to the unit circle causes the magnitude of the frequency response, at frequencies that correspond to points of the unit circle close to that point, to be small. In contrast, the presence of a pole close to the unit circle causes the magnitude of the frequency response to be large at frequencies close to that point. Thus poles have the opposite effect of zeros. Also, placing a zero close to a pole cancels the effect of the pole, and vice versa. This can be also seen from (5.2.12), since if $z_k = p_k$, the terms $e^{j\omega} - z_k$ and $e^{j\omega} - p_k$ cancel. Obviously, the presence of both poles and zeros in a transform results in a greater variety of shapes for $|H(\omega)|$ and $\measuredangle H(\omega)$. This observation is very important in the design of digital filters. We conclude our discussion with the following example illustrating these concepts.

EXAMPLE 5.2.2

Evaluate the frequency response of the system described by the system function

$$H(z) = \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

Solution. Clearly, $H(z)$ has a zero at $z = 0$ and a pole at $p = 0.8$. Hence the frequency response of the system is

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega} - 0.8}$$

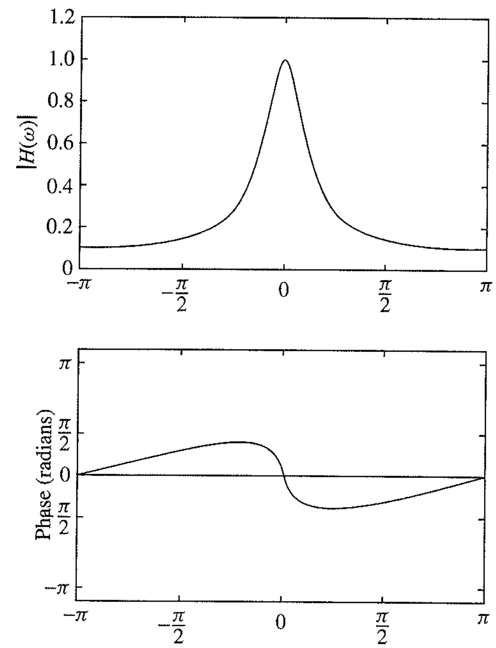

Figure 5.2.3 Magnitude and phase of system with $H(z) = 1/(1 - 0.8z^{-1})$.

The magnitude response is

$$|H(\omega)| = \frac{|e^{j\omega}|}{|e^{j\omega} - 0.8|} = \frac{1}{\sqrt{1.64 - 1.6\cos\omega}}$$

and the phase response is

$$\theta(\omega) = \omega - \tan^{-1}\frac{\sin\omega}{\cos\omega - 0.8}$$

The magnitude and phase responses are illustrated in Fig. 5.2.3. Note that the peak of the magnitude response occurs at $\omega = 0$, the point on the unit circle closest to the pole located at 0.8.

If the magnitude response in (5.2.17) is expressed in decibels,

$$|H(\omega)|_{dB} = 20\log_{10}|b_0| + 20\sum_{k=1}^{M}\log_{10}V_k(\omega) - 20\sum_{k=1}^{N}\log_{10}U_k(\omega) \qquad (5.2.23)$$

Thus the magnitude response is expressed as a sum of the magnitude factors in $|H(\omega)|$.

5.3 Correlation Functions and Spectra at the Output of LTI Systems

In this section, we derive the spectral relationships between the input and output signals of LTI systems. Section 5.3.1 describes the relationships for the energy density spectra of deterministic input and output signals. Section 5.3.2 is focused on the relationships for the power density spectra of random input and output signals.

5.3.1 Input–Output Correlation Functions and Spectra

In Section 2.6.4 we developed several correlation relationships between the input and the output sequences of an LTI system. Specifically, we derived the following equations:

$$r_{yy}(m) = r_{hh}(m) * r_{xx}(m) \tag{5.3.1}$$

$$r_{yx}(m) = h(m) * r_{xx}(m) \tag{5.3.2}$$

where $r_{xx}(m)$ is the autocorrelation sequence of the input signal $\{x(n)\}$, $r_{yy}(m)$ is the autocorrelation sequence of the output $\{y(n)\}$, $r_{hh}(m)$ is the autocorrelation sequence of the impulse response $\{h(n)\}$, and $r_{yx}(m)$ is the crosscorrelation sequence between the output and the input signals. Since (5.3.1) and (5.3.2) involve the convolution operation, the z-transform of these equations yields

$$\begin{aligned} S_{yy}(z) &= S_{hh}(z)S_{xx}(z) \\ &= H(z)H(z^{-1})S_{xx}(z) \end{aligned} \tag{5.3.3}$$

$$S_{yx}(z) = H(z)S_{xx}(z) \tag{5.3.4}$$

If we substitute $z = e^{j\omega}$ in (5.3.4), we obtain

$$\begin{aligned} S_{yx}(\omega) &= H(\omega)S_{xx}(\omega) \\ &= H(\omega)|X(\omega)|^2 \end{aligned} \tag{5.3.5}$$

where $S_{yx}(\omega)$ is the cross-energy density spectrum of $\{y(n)\}$ and $\{x(n)\}$. Similarly, evaluating $S_{yy}(z)$ on the unit circle yields the energy density spectrum of the output signal as

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \tag{5.3.6}$$

where $S_{xx}(\omega)$ is the energy density spectrum of the input signal.

Since $r_{yy}(m)$ and $S_{yy}(\omega)$ are a Fourier transform pair, it follows that

$$r_{yy}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)e^{j\omega m}\,d\omega \tag{5.3.7}$$

The total energy in the output signal is simply

$$\begin{aligned} E_y &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)d\omega = r_{yy}(0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2\, S_{xx}(\omega)d\omega \end{aligned} \tag{5.3.8}$$

The result in (5.3.8) may be used to easily prove that $E_y \geq 0$.

Finally, we note that if the input signal has a flat spectrum [i.e., $S_{xx}(\omega) = S_x =$ constant for $\pi \leq \omega \leq -\pi$], (5.3.5) reduces to

$$S_{yx}(\omega) = H(\omega)S_x \tag{5.3.9}$$

where S_x is the constant value of the spectrum. Hence

$$H(\omega) = \frac{1}{S_x} S_{yx}(\omega) \tag{5.3.10}$$

or, equivalently,

$$h(n) = \frac{1}{S_x} r_{yx}(m) \tag{5.3.11}$$

The relation in (5.3.11) implies that $h(n)$ can be determined by exciting the input to the system by a spectrally flat signal $\{x(n)\}$, and crosscorrelating the input with the output of the system. This method is useful in measuring the impulse response of an unknown system.

5.3.2 Correlation Functions and Power Spectra for Random Input Signals

This development parallels the derivations in Section 5.3.1, with the exception that we now deal with the statistical mean and autocorrelation of the input and output signals of an LTI system.

Let us consider a discrete-time linear time-invariant system with unit sample response $\{h(n)\}$ and frequency response $H(f)$. For this development we assume that $\{h(n)\}$ is real. Let $x(n)$ be a sample function of a stationary random process $X(n)$ that excites the system and let $y(n)$ denote the response of the system to $x(n)$.

From the convolution summation that relates the output to the input we have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \tag{5.3.12}$$

Since $x(n)$ is a random input signal, the output is also a random sequence. In other words, for each sample sequence $x(n)$ of the process $X(n)$, there is a corresponding sample sequence $y(n)$ of the output random process $Y(n)$. We wish to relate the statistical characteristics of the output random process $Y(n)$ to the statistical characterization of the input process and the characteristics of the system.

The expected value of the output $y(n)$ is

$$\begin{aligned} m_y \equiv E[y(n)] &= E[\sum_{k=-\infty}^{\infty} h(k)x(n-k)] \\ &= \sum_{k=-\infty}^{\infty} h(k)E[x(n-k)] \end{aligned} \tag{5.3.13}$$

$$m_y = m_x \sum_{k=-\infty}^{\infty} h(k)$$

From the Fourier transform relationship

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{5.3.14}$$

we have

$$H(0) = \sum_{k=-\infty}^{\infty} h(k) \tag{5.3.15}$$

which is the dc gain of the system. The relationship in (5.3.15) allows us to express the mean value in (5.3.13) as

$$m_y = m_x H(0) \tag{5.3.16}$$

The autocorrelation sequence for the output random process is defined as

$$\begin{aligned}
\gamma_{yy}(m) &= E[y^*(n)y(n+m)] \\
&= E\left[\sum_{k=-\infty}^{\infty} h(k)x^*(n-k) \sum_{j=-\infty}^{\infty} h(j)x(n+m-j)\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h(k)h(j)E[x^*(n-k)x(n+m-j)] \\
&= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h(k)h(j)\gamma_{xx}(k-j+m)
\end{aligned} \tag{5.3.17}$$

This is the general form for the autocorrelation of the output in terms of the autocorrelation of the input and the impulse response of the system.

A special form of (5.3.17) is obtained when the input random process is white, that is, when $m_x = 0$ and

$$\gamma_{xx}(m) = \sigma_x^2\delta(m) \tag{5.3.18}$$

where $\sigma_x^2 \equiv \gamma_{xx}(0)$ is the input signal power. Then (5.3.17) reduces to

$$\gamma_{yy}(m) = \sigma_x^2 \sum_{k=-\infty}^{\infty} h(k)h(k+m) \tag{5.3.19}$$

Under this condition the output process has the average power

$$\gamma_{yy}(0) = \sigma_x^2 \sum_{n=-\infty}^{\infty} h^2(n) = \sigma_x^2 \int_{-1/2}^{1/2} |H(f)|^2 df \tag{5.3.20}$$

where we have applied Parseval's theorem.