

pulse is expanded (compressed) in time, its transform is compressed (expanded) in frequency. This behavior, between the time function and its spectrum, is a type of uncertainty principle that appears in different forms in various branches of science and engineering.

Finally, the energy density spectrum of the rectangular pulse is

$$S_{xx}(F) = (A\tau)^2 \left( \frac{\sin \pi F\tau}{\pi F\tau} \right)^2 \quad (4.1.46)$$

## 4.2 Frequency Analysis of Discrete-Time Signals

In Section 4.1 we developed the Fourier series representation for continuous-time periodic (power) signals and the Fourier transform for finite energy aperiodic signals. In this section we repeat the development for the class of discrete-time signals.

As we have observed from the discussion of Section 4.1, the Fourier series representation of a continuous-time periodic signal can consist of an infinite number of frequency components, where the frequency spacing between two successive harmonically related frequencies is  $1/T_p$ , and where  $T_p$  is the fundamental period. Since the frequency range for continuous-time signals extends from  $-\infty$  to  $\infty$ , it is possible to have signals that contain an infinite number of frequency components. In contrast, the frequency range for discrete-time signals is unique over the interval  $(-\pi, \pi)$  or  $(0, 2\pi)$ . A discrete-time signal of fundamental period  $N$  can consist of frequency components separated by  $2\pi/N$  radians or  $f = 1/N$  cycles. Consequently, the Fourier series representation of the discrete-time periodic signal will contain at most  $N$  frequency components. This is the basic difference between the Fourier series representations for continuous-time and discrete-time periodic signals.

### 4.2.1 The Fourier Series for Discrete-Time Periodic Signals

Suppose that we are given a periodic sequence  $x(n)$  with period  $N$ , that is,  $x(n) = x(n + N)$  for all  $n$ . The Fourier series representation for  $x(n)$  consists of  $N$  harmonically related exponential functions

$$e^{j2\pi kn/N}, \quad k = 0, 1, \dots, N - 1$$

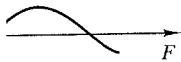
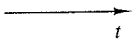
and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad (4.2.1)$$

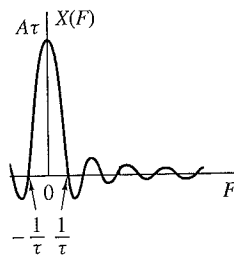
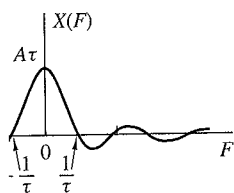
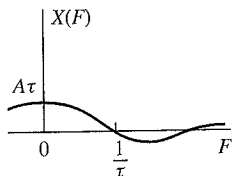
where the  $\{c_k\}$  are the coefficients in the series representation.

To derive the expression for the Fourier coefficients, we use the following formula:

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases} \quad (4.2.2)$$



(narrower) and more  
1.9. Thus as the signal



Note the similarity of (4.2.2) with the continuous-time counterpart in (4.1.3). The proof of (4.2.2) follows immediately from the application of the geometric summation formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \frac{1-a^N}{1-a}, & a \neq 1 \end{cases} \quad (4.2.3)$$

The expression for the Fourier coefficients  $c_k$  can be obtained by multiplying both sides of (4.2.1) by the exponential  $e^{-j2\pi ln/N}$  and summing the product from  $n = 0$  to  $n = N - 1$ . Thus

$$\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j2\pi(k-l)n/N} \quad (4.2.4)$$

If we perform the summation over  $n$  first, in the right-hand side of (4.2.4), we obtain

$$\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \begin{cases} N, & k-l = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases} \quad (4.2.5)$$

where we have made use of (4.2.2). Therefore, the right-hand side of (4.2.4) reduces to  $Nc_l$  and hence

$$c_l = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N}, \quad l = 0, 1, \dots, N-1 \quad (4.2.6)$$

Thus we have the desired expression for the Fourier coefficients in terms of the signal  $x(n)$ .

The relationships (4.2.1) and (4.2.6) for the frequency analysis of discrete-time signals are summarized below.

Frequency Analysis of Discrete-Time Periodic Signals

Synthesis equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad (4.2.7)$
Analysis equation	$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad (4.2.8)$

Equation (4.2.7) is often called the *discrete-time Fourier series* (DTFS). The Fourier coefficients  $\{c_k\}$ ,  $k = 0, 1, \dots, N - 1$  provide the description of  $x(n)$  in the frequency domain, in the sense that  $c_k$  represents the amplitude and phase associated with the frequency component

$$s_k(n) = e^{j2\pi kn/N} = e^{j\omega_k n}$$

where  $\omega_k = 2\pi k/N$ .

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We recall from Section 1.3.3 that the functions  $s_k(n)$  are periodic with period  $N$ . Hence  $s_k(n) = s_k(n + N)$ . In view of this periodicity, it follows that the Fourier coefficients  $c_k$ , when viewed beyond the range  $k = 0, 1, \dots, N - 1$ , also satisfy a periodicity condition. Indeed, from (4.2.8), which holds for every value of  $k$ , we have

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = c_k \quad (4.2.9)$$

Therefore, the Fourier series coefficients  $\{c_k\}$  form a periodic sequence when extended outside of the range  $k = 0, 1, \dots, N - 1$ . Hence

$$c_{k+N} = c_k$$

that is,  $\{c_k\}$  is a periodic sequence with fundamental period  $N$ . Thus the spectrum of a signal  $x(n)$ , which is periodic with period  $N$ , is a periodic sequence with period  $N$ . Consequently, any  $N$  consecutive samples of the signal or its spectrum provide a complete description of the signal in the time or frequency domains.

Although the Fourier coefficients form a periodic sequence, we will focus our attention on the single period with range  $k = 0, 1, \dots, N - 1$ . This is convenient, since in the frequency domain, this amounts to covering the fundamental range  $0 \leq \omega_k = 2\pi k/N < 2\pi$ , for  $0 \leq k \leq N - 1$ . In contrast, the frequency range  $-\pi < \omega_k = 2\pi k/N \leq \pi$  corresponds to  $-N/2 < k \leq N/2$ , which creates an inconvenience when  $N$  is odd. Clearly, if we use a sampling frequency  $F_s$ , the range  $0 \leq k \leq N - 1$  corresponds to the frequency range  $0 \leq F < F_s$ .

#### EXAMPLE 4.2.1

Determine the spectra of the signals

- (a)  $x(n) = \cos \sqrt{2}\pi n$
- (b)  $x(n) = \cos \pi n/3$
- (c)  $x(n)$  is periodic with period  $N = 4$  and  $x(n) = \{1, 1, 0, 0\}$

#### Solution.

- (a) For  $\omega_0 = \sqrt{2}\pi$ , we have  $f_0 = 1/\sqrt{2}$ . Since  $f_0$  is not a rational number, the signal is not periodic. Consequently, this signal cannot be expanded in a Fourier series. Nevertheless, the signal does possess a spectrum. Its spectral content consists of the single frequency component at  $\omega = \omega_0 = \sqrt{2}\pi$ .
- (b) In this case  $f_0 = \frac{1}{6}$  and hence  $x(n)$  is periodic with fundamental period  $N = 6$ . From (4.2.8) we have

$$c_k = \frac{1}{6} \sum_{n=0}^5 x(n) e^{-j2\pi kn/6}, \quad k = 0, 1, \dots, 5$$