pulse is expanded (compressed) in time, its transform is compressed (expanded) in frequency. This behavior, between the time function and its spectrum, is a type of uncertainty principle that appears in different forms in various branches of science and engineering.

Finally, the energy density spectrum of the rectangular pulse is

$$S_{xx}(F) = (A\tau)^2 \left(\frac{\sin \pi F\tau}{\pi F\tau}\right)^2 \tag{4.1.46}$$

4.2 Frequency Analysis of Discrete-Time Signals

In Section 4.1 we developed the Fourier series representation for continuous-time periodic (power) signals and the Fourier transform for finite energy aperiodic signals. In this section we repeat the development for the class of discrete-time signals.

As we have observed from the discussion of Section 4.1, the Fourier series representation of a continuous-time periodic signal can consist of an infinite number of frequency components, where the frequency spacing between two successive harmonically related frequencies is $1/T_p$, and where T_p is the fundamental period. Since the frequency range for continuous-time signals extends from $-\infty$ to ∞, it is possible to have signals that contain an infinite number of frequency components. In contrast, the frequency range for discrete-time signals is unique over the interval $(-\pi, \pi)$ or $(0, 2\pi)$. A discrete-time signal of fundamental period N can consist of frequency components separated by $2\pi/N$ radians or $f = 1/N$ cycles. Consequently, the Fourier series representation of the discrete-time periodic signal will contain at most N frequency components. This is the basic difference between the Fourier series representations for continuous-time and discrete-time periodic signals.

4.2.1 The Fourier Series for Discrete-Time Periodic Signals

Suppose that we are given a periodic sequence $x(n)$ with period N, that is, $x(n) = x(n + N)$ for all n. The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{4.2.1}$$

where the $\{c_k\}$ are the coefficients in the series representation.

To derive the expression for the Fourier coefficients, we use the following formula:

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.2}$$

Note the similarity of (4.2.2) with the continuous-time counterpart in (4.1.3). The proof of (4.2.2) follows immediately from the application of the geometric summation formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \tag{4.2.3}$$

The expression for the Fourier coefficients c_k can be obtained by multiplying both sides of (4.2.1) by the exponential $e^{-j2\pi ln/N}$ and summing the product from $n = 0$ to $n = N - 1$. Thus

$$\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{j2\pi(k-l)n/N} \tag{4.2.4}$$

If we perform the summation over n first, in the right-hand side of (4.2.4), we obtain

$$\sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \begin{cases} N, & k - l = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.5}$$

where we have made use of (4.2.2). Therefore, the right-hand side of (4.2.4) reduces to Nc_l and hence

$$c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N}, \qquad l = 0, 1, \ldots, N-1 \tag{4.2.6}$$

Thus we have the desired expression for the Fourier coefficients in terms of the signal $x(n)$.

The relationships (4.2.1) and (4.2.6) for the frequency analysis of discrete-time signals are summarized below.

Frequency Analysis of Discrete-Time Periodic Signals

Synthesis equation	$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$	(4.2.7)
Analysis equation	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$	(4.2.8)

Equation (4.2.7) is often called the *discrete-time Fourier series* (DTFS). The Fourier coefficients $\{c_k\}$, $k = 0, 1, \ldots, N-1$ provide the description of $x(n)$ in the frequency domain, in the sense that c_k represents the amplitude and phase associated with the frequency component

$$s_k(n) = e^{j2\pi kn/N} = e^{j\omega_k n}$$

where $\omega_k = 2\pi k/N$.

We recall from Section 1.3.3 that the functions $s_k(n)$ are periodic with period N. Hence $s_k(n) = s_k(n+N)$. In view of this periodicity, it follows that the Fourier coefficients c_k, when viewed beyond the range $k = 0, 1, \ldots, N-1$, also satisfy a periodicity condition. Indeed, from (4.2.8), which holds for every value of k, we have

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi(k+N)n/N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = c_k \qquad (4.2.9)$$

Therefore, the Fourier series coefficients $\{c_k\}$ form a periodic sequence when extended outside of the range $k = 0, 1, \ldots, N-1$. Hence

$$c_{k+N} = c_k$$

that is, $\{c_k\}$ is a periodic sequence with fundamental period N. *Thus the spectrum of a signal $x(n)$, which is periodic with period N, is a periodic sequence with period N.* Consequently, any N consecutive samples of the signal or its spectrum provide a complete description of the signal in the time or frequency domains.

Although the Fourier coefficients form a periodic sequence, we will focus our attention on the single period with range $k = 0, 1, \ldots, N-1$. This is convenient, since in the frequency domain, this amounts to covering the fundamental range $0 \le \omega_k = 2\pi k/N < 2\pi$, for $0 \le k \le N-1$. In contrast, the frequency range $-\pi < \omega_k = 2\pi k/N \le \pi$ corresponds to $-N/2 < k \le N/2$, which creates an inconvenience when N is odd. Clearly, if we use a sampling frequency F_s, the range $0 \le k \le N-1$ corresponds to the frequency range $0 \le F < F_s$.

EXAMPLE 4.2.1

Determine the spectra of the signals

(a) $x(n) = \cos\sqrt{2}\pi n$

(b) $x(n) = \cos \pi n/3$

(c) $x(n)$ is periodic with period $N = 4$ and $x(n) = \{\underset{\uparrow}{1}, 1, 0, 0\}$

Solution.

(a) For $\omega_0 = \sqrt{2}\pi$, we have $f_0 = 1/\sqrt{2}$. Since f_0 is not a rational number, the signal is not periodic. Consequently, this signal cannot be expanded in a Fourier series. Nevertheless, the signal does possess a spectrum. Its spectral content consists of the single frequency component at $\omega = \omega_0 = \sqrt{2}\pi$.

(b) In this case $f_0 = \frac{1}{6}$ and hence $x(n)$ is periodic with fundamental period $N = 6$. From (4.2.8) we have

$$c_k = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j2\pi kn/6}, \qquad k = 0, 1, \ldots, 5$$