Continuous-Time Fourier Transform: (CTFT)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$$

Inverse CTFT:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$$

Discrete-Time Fourier Transform: (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse DTFT:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

where

$$G(f_n) = \int_{-T_0/2}^{T_0/2} g_p(t)\exp(-j2\pi f_n t)dt \tag{1.27}$$

Suppose we now let the period T_0 approach infinity, or equivalently its reciprocal Δf approach zero. Then, we find that, in the limit, the discrete frequency f_n approaches the continuous frequency variable f, and the discrete sum in Eq. (1.26) becomes an integral defining the area under a continuous function of frequency f, namely, $G(f)\exp(j2\pi ft)$. Also, as T_0 approaches infinity, the function $g_p(t)$ approaches $g(t)$. Therefore, in the limit, Eqs. (1.26) and (1.27) become, respectively,

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df \tag{1.28}$$

where

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt \tag{1.29}$$

We have thus achieved our aim of representing an arbitrarily defined signal $g(t)$ in terms of exponential functions over the entire interval $(-\infty < t < \infty)$. Note that in Eqs. (1.28) and (1.29) we have used a lowercase letter to denote the time function and an uppercase letter to denote the corresponding frequency function.

Equation (1.29) states that, given a time function $g(t)$, we can determine a new function $G(f)$ of the frequency variable f. Equation (1.28) states that, given this new or transformed function $G(f)$, we can recover the original time function $g(t)$. Thus, since from $g(t)$ we can define the function $G(f)$ and from $G(f)$ we can reconstruct $g(t)$, the time function is also specified by $G(f)$. The function $G(f)$ can be thought of as a transformed version of $g(t)$ and is referred to as the *Fourier transform* of $g(t)$. The time function $g(t)$ is similarly referred to as the *inverse Fourier transform* of $G(f)$. The functions $g(t)$ and $G(f)$ are said to constitute a *Fourier transform pair*, and one is called the *mate* of the other.*

For a signal $g(t)$ to be Fourier transformable, it is sufficient that $g(t)$ satisfies Dirichlet's conditions:

1. The function $g(t)$ is single-valued, with a finite number of maxima and minima and a finite number of discontinuities in any finite time interval.
2. The function $g(t)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |g(t)|dt < \infty$$

* For an extensive list of Fourier transform pairs, see G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications* (Van Nostrand, 1948).

Property 1 Linearity (Superposition)

Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then for all constants a and b, we have

$$a g_1(t) + b g_2(t) \rightleftharpoons a G_1(f) + b G_2(f) \tag{1.41}$$

The proof of this property follows simply from the linearity of the integrals defining $G(f)$ and $g(t)$.

Table 1.1 Summary of Properties of the Fourier Transform

Property	Mathematical Description
1. Linearity	$a g_1(t) + b g_2(t) \rightleftharpoons a G_1(f) + b G_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\,dt = G(0)$
7. Area under $G(f)$.	$g(0) = \int_{-\infty}^{\infty} G(f)\,df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Differentiation in the frequency domain	$-j 2 \pi t \longleftrightarrow \frac{d G(f)}{df}$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\,d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau)\,d\tau \rightleftharpoons G_1(f) G_2(f)$.

Parseval's Theorem — area under $|g(t)|^2$ = area under $|G(f)|^2$

Let $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. *Then*

$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \rightleftharpoons G_1(f) G_2(f) \tag{1.89}$$

This result follows directly by combining Property 3 (duality) and Property 11 (time-domain multiplication). We may thus state that *the convolution of two signals in the time domain is transformed into the multiplication of their individual Fourier transforms in the frequency domain.*

Example 17

Consider the convolution of the rectangular pulse $A\,\text{rect}(t/T)$ with itself. As illustrated in Fig. 1.17, the result of this convolution process is a triangular pulse of duration $2T$ and amplitude A^2T, centered at $t = 0$. Because the Fourier transform of the rectangular pulse is equal to $AT\,\text{sinc}(fT)$, it follows from Property 12 that the Fourier transform of the triangular pulse of Fig. 1.17(c) is equal to $A^2T^2\,\text{sinc}^2(fT)$. Except for a change in the scaling factor, this result is exactly the same as that obtained in Example 14.

In Table 1.2 we have collected for reference a number of basic Fourier transform pairs derived in this section and the previous one.

Table 1.2 Fourier Transform Pairs.

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a > 0$	$\frac{1}{a + j2\pi f}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1 - \frac{\lvert t \rvert}{T}, & \lvert t \rvert < T \\ 0, & \lvert t \rvert \geq T \end{cases}$	$T\,\text{sinc}^2(fT)$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) \rightleftharpoons \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \quad (1$$

Equation (1.131) states that the Fourier transform of a periodic train of delta funct spaced T_0 seconds apart, consists of another set of delta functions scaled by the factor $1/T_{\cdot}$ regularly spaced $1/T_0$ hertz apart along the frequency axis as in Fig. 1.30(b). In the special of the period T_0 equal to 1 second, a periodic train of delta functions is, like a Gaussian p its own transform.

In Table 1.3, we have collected for reference the Fourier transforms of the vari finite-power signals considered in this section.

Table 1.3 Fourier Transforms of Finite-Power Signals

Time Function	Fourier Transform
$\delta(t)$	1
1	$\delta(f)$
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$	$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$