Connection Between Cross-Correlation & Matched Filtering

$$z[n] = r_{yx}[n] = y[n] * \underbrace{x^*[-n]}_{h_{MF}[n]}$$

$=$ running matched filter output

$=$ cross-correlation between $y[n]$ and $x[n]$

If: $$y[n] = \sum_{k=1}^{K} a_k \, x[n-D_k]$$

Then: $$r_{yx}[n] = \left\{ \sum_{k=1}^{K} a_k \, x[n-D_k] \right\} * x^*[-n]$$

$$= \sum_{k=1}^{K} a_k \, r_{xx}[n-D_k]$$

Where: $r_{xx}[n] = x[n] * x^*[-n] =$ auto-correlation

Since: $x[n-D_k] * x^*[-n] = r_{xx}[n-D_k]$

Autocorrelation Properties as a Function of n

$$\boxed{r_{xx}[n] = x[n] * x^*[-n]}$$

1. $r_{xx}[-n] = r_{xx}^*[n]$

2. $|r_{xx}[n]| < r_{xx}[0] \quad \forall n$

3. $\sum_{n=-\infty}^{\infty} r_{xx}[n] e^{-j\omega n} \geq 0$ for all ω (also real-valued $\forall \omega$)

4. $x[n]$ and $x[n-n_0]$ have same autocorrelation function

5. $y[n] = e^{j(\omega_0 n + \theta)} x[n]$

   $\Rightarrow r_{yy}[n] = e^{j\omega_0 n} r_{xx}[n]$

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{y[n]}$$

any LTI System

$r_{yy}[n] = r_{xx}[n] * r_{hh}[n]$

$r_{yx}[n] = r_{xx}[n] * h[n]$

cross-correlation between input and output

Sect. 2.6.2 Properties of Autocorrelation and Cross-Correlation Sequences ①

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-\ell]$$

1. $|r_{xx}[\ell]| \leq r_{xx}[0]$ for all ℓ

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x = \text{energy}$$

2. $$\rho_{xx}[\ell] = \frac{r_{xx}[\ell]}{r_{xx}[0]}$$

$$|\rho_{xx}[\ell]| \leq 1 \quad \forall \ell$$

Proof: Tricky! (2)

$$\sum_{n=-\infty}^{\infty} \left\{ z\, x[n] + y[n-\ell] \right\}^2 \geq 0$$

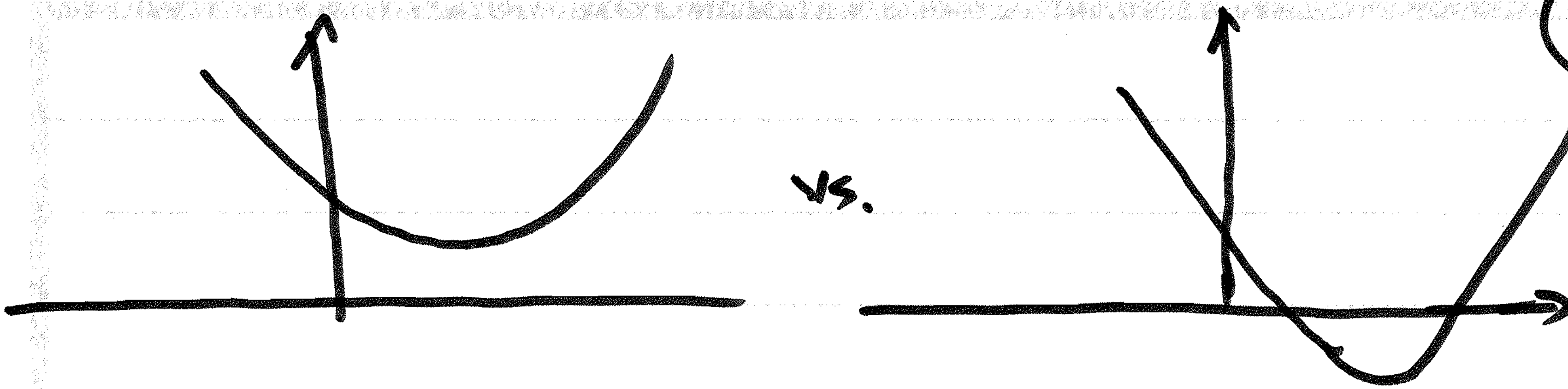
$$= z^2 \sum_{n=-\infty}^{\infty} x^2[n] + 2z \sum_{n=-\infty}^{\infty} x[n]\, y[n-\ell]$$

$$+ \sum_{n=-\infty}^{\infty} y^2[n-\ell]$$

$$= z^2\, r_{xx}[0] + 2z\, r_{xy}[\ell] + r_{yy}[0] \geq 0$$

viewed as polynomial in z that can never go negative

$\Rightarrow$ must have complex-valued roots



Thus, discriminant must be such that $b^2 - 4ac < 0$

$$a z^2 + b z + c \quad \left\{ \begin{array}{l} a = r_{xx}[0] \\ b = 2\, r_{xy}[\ell] \\ c = r_{yy}[0] \end{array} \right.$$

THUS:

$$b^2 - 4ac = 4\, r_{xy}^2[\ell] - 4\, r_{xx}[0]\, r_{yy}[0] \le 0$$

$$r_{xy}^2[\ell] \le r_{xx}[0]\, r_{yy}[0]$$

$$|r_{xy}[\ell]| \le \sqrt{r_{xx}[0]\, r_{yy}[0]}$$

If $y[n] = x[n]$, then:

$$|r_{xx}[\ell]| \leq \sqrt{r_{xx}^2[0]}$$

$$|r_{xx}[\ell]| \leq r_{xx}[0]$$

Also, follows that with

$$\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}}$$

then

$$|\rho_{xy}[\ell]| \leq 1 \quad \forall \ell$$

Summarizing three main properties of autocorrelation sequence $r_{xx}[\ell] = x[\ell] * x^*[-\ell]$

1. $r_{xx}[-\ell] = r_{xx}^*[\ell]$

2. $|r_{xx}[\ell]| \le r_{xx}[0]$

3. $\sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell} \ge 0$ for all ω

From 1., $S_{xx}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell}$

is real-valued for all ω

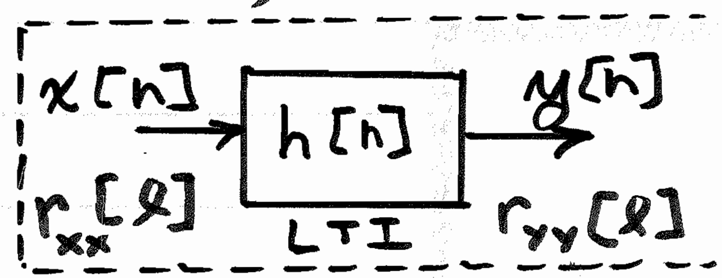
$S_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$

$\Rightarrow$ energy density spectrum

I/O Relationships for $r_{xx}[\ell]$, $r_{yx}[\ell]$, and $r_{yy}[\ell]$

Recall: $r_{xx}[\ell] = x[\ell] * x^*[-\ell]$

$$
\begin{aligned}
r_{yx}[\ell] &= y[\ell] * x^*[-\ell] \\
&= \left(h[\ell] * x[\ell]\right) * x^*[-\ell] \\
&= h[\ell] * \left(x[\ell] * x^*[-\ell]\right) \\
&= h[\ell] * r_{xx}[\ell]
\end{aligned}
$$



Similarly:

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

Ideal Radar Problem Revisited

- <u>recall</u>: $x[n] * \delta[n-n_0] = x[n-n_0]$

- single target with no noise

$$y[n] = \Gamma\, x[n-D]$$

$$= x[n] * \underbrace{\Gamma\, \delta[n-D]}_{h[n] = \Gamma \delta[n-D]}$$

- can model as LTI system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = \Gamma\, \delta[n-D]$$

We know: $r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$

- Thus: $r_{yx}[\ell] = \Gamma\, r_{xx}[\ell - D]$

Two-Target Case :

$$y[n] = T_1\, x[n-D_1] + T_2\, x[n-D_2]$$

$$= x[n] * \{T_1\, \delta[n-D_1] + T_2\, \delta[n-D_2]\}$$

$$= x[n] * h[n]$$

Just a reminder: $x[n]$ is the transmitted signal, and we cross-correlate it with the received signal $y[n] \Rightarrow r_{yx}[\ell] = y[\ell] * x^*[-\ell]$ and look for peaks $\Rightarrow$ target round-trip delays

From previous result, we have:

$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$$

$$= T_1\, r_{xx}[\ell-D_1] + T_2\, r_{xx}[\ell-D_2]$$

Desire autocorrelation sequence that approximates a Kronecker Delta $\delta[n]$, especially to resolve closely-spaced targets

- else $r_{xx}[\ell - D_1]$ overlaps with $r_{xx}[\ell - D_2]$
- could yield only a single peak at some delay between D_1 and D_2
- or weaker (smaller) target masked by the "sidelobes" of the stronger target
- desire "constant modulus" signals, e.g. sequence of "+1"s and "-1"s, since for power amplifier efficiency, we operate in nonlinear region

Example 2.6.2

$x[n] = a^n u[n]$

a = real-valued

$|a| < 1$

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-\ell]$$

since: $r_{xx}[-\ell] = r_{xx}[\ell]$, let's compute

$x[n]$ real

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[n+\ell]$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n]\, a^{n+\ell} u[n+\ell]$$

turns on $n=0$

turns on at $n=-\ell$

$$= \sum_{n=0}^{\infty} a^n a^{n+\ell} = a^\ell \sum_{n=0}^{\infty} a^{2n} = a^\ell \sum_{n=0}^{\infty} (a^2)^n$$

$$= \frac{1}{1-a^2} a^\ell \implies \boxed{r_{xx}[\ell] = \frac{1}{1-a^2} a^{|\ell|}}$$

Final note on Cross-Correlation

$$r_{yx}[\ell] = y[\ell] * x^*[-\ell]$$

or $$r_{yx}[m] = y[m] * x^*[-m]$$

Sidenote: Chap. 12 uses "m" for lag variable in stochastic version of auto-correlat
I will use "ℓ" and "m" interchangeably

- Any time you convolve two DT sequences, one can be viewed as an input signal and the other can be viewed as a filter impulse response

$$y[m] \longrightarrow \underbrace{\boxed{h[m] = x^*[-m]}}_{\text{"matched filter"}} \longrightarrow r_{yx}[m]$$

$\Rightarrow$ "matched" to transmitted signal

$\Rightarrow$ a running cross-correlator is a matched filter

Some additional properties of Auto-Correlation and Cross-Correlation

- First, note: $\delta[n-n_1] * \delta[n-n_2]$
$$= \delta[n-(n_1+n_2)]$$
- In particular, if $n_2 = -n_1$:
$$\delta[n-n_1] * \delta[n+n_1] = \delta[n]$$

- **Show:** $x[n] \Rightarrow$ autocorrelation is $r_{xx}[\ell]$

$x[n]$ and $x[n-n_0]$ have same autocorrelation

- recall: $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

where:
$r_{hh}[\ell] = h[\ell] * h^*[-\ell]$

$$r_{yy}[\ell] = r_{hh}[\ell] * r_{xx}[\ell]$$

• Consider: $y[n] = x[n-n_0] = x[n] * \delta[n-n_0]$

$$h[n] = \delta[n-n_0]$$

Thus:

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

where: $r_{hh}[\ell] = \delta[\ell-n_0] * \delta[-\ell-n_0]$

delta fn, is symmetric

$$= \delta[\ell-n_0] * \delta[\ell+n_0]$$

$$= \delta[\ell]$$

Thus: $r_{yy}[\ell] = r_{xx}[\ell] * \delta[\ell] = r_{xx}[\ell]$

• A time-shift does <u>not</u> change <u>auto</u>-correlation

• See Prob. 1, part (c) from Exam 1 for Fall 2006

• Solution for exam used frequency domain to prove same result (and used "m" instead of "ℓ")

- **Another result:**

If: $y[n] = e^{j(\omega_0 n + \theta)} x[n]$

Then: $r_{yy}[\ell] = e^{j\omega_0 \ell} r_{xx}[\ell]$

$$r_{yy}[\ell] = y[\ell] * y^*[-\ell] = y^*[-\ell] * y[\ell]$$

minus sign

$$= e^{-j(\omega_0(-\ell)+\theta)} x^*[-\ell] * e^{j(\omega_0 \ell + \theta)} x[\ell]$$

$$= e^{-j\theta} e^{j\theta} \; e^{j\omega_0 \ell} x^*[-\ell] * e^{j\omega_0 \ell} x[\ell]$$

$$= \sum_{k=-\infty}^{\infty} e^{j\omega_0 k} x^*[-k] \; e^{j\omega_0(\ell-k)} x[\ell-k]$$

$$= e^{j\omega_0 \ell} \sum_{k=-\infty}^{\infty} x^*[-k] x[\ell-k] = e^{j\omega_0 \ell}\left(x^*[-\ell] * x[\ell]\right)$$

$$= e^{j\omega_0 \ell} r_{xx}[\ell]$$