Connection Between Cross-Correlation and Matched Filtering

\[ z[n] = r_{yx}[n] = y[n] \ast x^*[-n] \]

\[ = \text{running matched filter output} \]
\[ = \text{cross-correlation between } y[n] \text{ and } x[n] \]

If: \[ y[n] = \sum_{k=1}^{K} a_k x[n-D_k] \]

Then: \[ r_{yx}[n] = \left\{ \sum_{k=1}^{K} a_k x[n-D_k] \right\} \ast x^*[-n] \]
\[ = \sum_{k=1}^{K} a_k r_{xx}[n-D_k] \]

Where: \[ r_{xx}[n] = x[n] \ast x^*[-n] = \text{auto-correlation} \]

Since: \[ x[n-D_k] \ast x^*[-n] = r_{xx}[n-D_k] \]
Auto correlation Properties as a Function of $n$

1. $r_{xx}[-n] = r_{xx}^*[n]$

2. $|r_{xx}[n]| < r_{xx}[0] + n$

3. $\sum_{n=-\infty}^{\infty} r_{xx}[n] e^{-j\omega n} \geq 0$ for all $\omega$ (also real-valued $+\omega$)

4. $x[n]$ and $x[n-n_0]$ have same auto-correlation function

5. $y[n] = e^{j(\omega_0 n + \theta)} x[n]$
   $\Rightarrow r_{yy}[n] = e^{j\omega_0 n} r_{xx}[n]$

- $\mathbf{x}[n] \rightarrow h[n] \rightarrow y[n]$
- $r_{yy}[n] = r_{xx}[n] * r_{hh}[n]$
- $r_{yx}[n] = r_{xx}[n] * h[n]$
- Any LTI System
- Cross-correlation between input and output
Sect. 2.6.2 Properties of \( \mathcal{O} \)

Autocorrelation and Cross-Correlation Sequences

\[
\rho_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n] x[n-l]
\]

1. \( |\rho_{xx}[l]| \leq \rho_{xx}[0] \) for all \( l \)

\[
\rho_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x = \text{energy}
\]

2. \( \rho_{xx}[l] = \frac{\rho_{xx}[l]}{\rho_{xx}[0]} \)

\[
|\rho_{xx}[l]| \leq 1 + \rho
\]
Proof: Tricky!

\[ \sum_{n=-\infty}^{\infty} \{z \times x[n] + y[n-l]\}^2 \geq 0 \]

\[ = z^2 \sum_{n=-\infty}^{\infty} x^2[n] + 2z \sum_{n=-\infty}^{\infty} x[n]y[n-l] + \sum_{n=-\infty}^{\infty} y^2[n-l] \]

\[ = z^2 r_{xx}[0] + 2z r_{xy}[l] + r_{yy}[0] \geq 0 \]

viewed as polynomial in \( z \) that can never go negative

\( \Rightarrow \) must have complex-valued roots
Thus, discriminant must be such that $b^2 - 4ac < 0$

$$a z^2 + b z + c \begin{cases} a = r_{xx}[0] \\ b = 2 r_{xy}[l] \\ c = r_{yy}[0] \end{cases}$$

Thus:

$$b^2 - 4ac = 4 r_{xy}^2[l] - 4 r_{xx}[0] r_{yy}[0] \leq 0$$

$$r_{xy}^2[l] \leq r_{xx}[0] r_{yy}[0]$$

$$| r_{xy}[l] | \leq \sqrt{r_{xx}[0] r_{yy}[0]}$$
If $y(n) = x(n)$, then:

\[ |r_{xx}[l]| \leq \sqrt{r_{xx}[0]} \]

Also, follows that with

\[ p_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \]

then

\[ |p_{xy}[l]| \leq 1 + \varepsilon \]
Summarizing three main properties of auto correlation sequence \( r_{xx}[l] = x[l] \ast x^*[−l] \)

1. \( r_{xx}[-l] = r_{xx}^*[l] \)

2. \( |r_{xx}[l]| \leq r_{xx}[0] \)

3. \( \sum_{l=-\infty}^{\infty} r_{xx}[l] e^{-j\omega l} \geq 0 \)
   for all \( \omega \)

From 1., \( S_{xx}(\omega) = \sum_{l=-\infty}^{\infty} r_{xx}[l] e^{-j\omega l} \)

is real-valued for all \( \omega \)

\( S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2 \)

\( => \) energy density spectrum
I/O Relationships for $r_{xx}[l]$, $r_{yx}[l]$, and $r_{yy}[l]$

Recall: $r_{xx}[l] = x[l] * x^*[-l]$

$r_{yx}[l] = y[l] * x^*[-l]$

$= (h[l] * x[l]) * x^*[-l]$

$= h[l] * (x[l] * x^*[-l])$

$= h[l] * r_{xx}[l]$

Similarly:

$r_{yy}[l] = r_{xx}[l] * r_{hh}[l]$
Ideal Radar Problem Revisited

Recall: $x[n] \ast \delta[n-n_0] = x[n-n_0]

Single target with no noise

$y[n] = T \times x[n-D]$

$= x[n] \ast \Gamma \delta[n-D]$

$h[n] = T \delta[n-D]$

Can model as LTI system:

$X[n] \rightarrow H[n] \rightarrow Y[n]$

$h[n] = T \delta[n-D]$

(We know: $r_{yx}[l] = r_{xx}[l] \ast h[l]$

Thus: $r_{yx}[l] = \Gamma r_{xx}[l-D]$
Two-Target Case:

\[ y[n] = T_1 x[n-D_1] + T_2 x[n-D_2] \]

\[ = x[n] * \left\{ T_1 \delta[n-D_1] + T_2 \delta[n-D_2] \right\} \]

\[ = x[n] * h[n] \]

Just a reminder: \( x[n] \) is the transmitted signal, and we cross-correlate it with the received signal \( y[n] \Rightarrow r_{yx}[l] = y[l] * x^*[l] \)

and look for peaks \( \Rightarrow \) target round-trip delays

From previous result, we have:

\[ r_{yx}[l] = r_{xx}[l] * h[l] \]

\[ = T_1 r_{xx}[l-D_1] + T_2 r_{xx}[l-D_2] \]
Desire autocorrelation sequence that approximates a Kronecker Delta $\delta[n]$, especially to resolve closely-spaced targets.

- $r_{xx}[\ell-D_1]$ overlaps with $r_{xx}[\ell-D_2]$

- Could yield only a single peak at some delay between $D_1$ and $D_2$

- Or weaker (smaller) target masked by the "sidelobes" of the stronger target

- Desire "constant modulus" signals, e.g., sequence of "$+1"$'s and "$-1"$'s, since for power amplifier efficiency, we operate in nonlinear region
Example 2.6.2 \[ x[n] = a^n u[n] \]
\[ a = \text{real-valued} \]

\[ r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n] x[n-l] \]
\[ |a| < 1 \]

Since \( r_{xx}[-l] = r_{xx}[l] \), let's compute \( x[n] \) real

\[ r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n] x[n+l] \]

\[ = \sum_{n=-\infty}^{\infty} a^n u[n] a^{n+l} u[n+l] \]

\[ = \sum_{n=0}^{\infty} a^n a^{n+l} = a^l \sum_{n=0}^{\infty} a^{2n} = a^l \sum_{n=0}^{\infty} (a^2)^n \]

\[ = \frac{1}{1-a^2} a^l \]

\[ \Rightarrow \]

\[ r_{xx}[l] = \frac{1}{1-a^2} a^{1+l} \]
Final note on Cross-Correlation

\[ r_{yx}(\ell) = y[\ell] \ast x^*(-\ell) \]

\[ r_{yx}[m] = y[m] \ast x^*(-m) \]

Sidenote: Chap. 12 uses "m" for lag variable in stochastic version of auto-correlation.

I will use "\( \ell \) and "m" interchangeably.

Any time you convolve two DT sequences, one can be viewed as an input signal and the other can be viewed as a filter impulse response.

\[ y[m] \rightarrow h[m] = x^*(-m) \rightarrow r_{yx}[m] \]

"matched filter"

⇒ "matched" to transmitted signal

⇒ a running cross-correlator is a matched filter
Some additional properties of Auto-Correlation and Cross-Correlation

1. First, note: $\delta[n-n_1] * \delta[n-n_2] = \delta[n-(n_1+n_2)]$

2. In particular, if $n_2 = -n_1$:
   
   $\delta[n-n_1] * \delta[n+n_1] = \delta[n]$

3. Show: $x[n] \Rightarrow$ auto-correlation is $r_{xx}[l]$
   
   $x[n]$ and $x[n-n_0]$ have same auto-correlation

4. Recall: $x[n] \rightarrow h[n] \rightarrow y[n]$
   
   where:
   
   $r_{hh}[l] = h[l] * h^*[l-\ell]$
   
   $r_{yy}[l] = r_{hh}[l] * r_{xx}[l]$
Consider: \( y[n] = x[n-n_0] = x[n] * \delta[n-n_0] \)

Thus:

\[ r_{yy}[l] = r_{xx}[l] * h[l] \]

where \( h[l] = \delta[l-n_0] \)

\[ r_{hh}[l] = \delta[l-n_0] * \delta[-l-n_0] \]

Delta fn, is symmetric

\[ = \delta[l-n_0] * \delta[l+n_0] \]

Thus:

\[ r_{yy}[l] = r_{xx}[l] * \delta[l] = r_{xx}[l] \]

A time-shift does not change auto-correlation

See Prob. 1, part (c) from Exam 1 for Fall 2006

Solution for exam used frequency domain to prove (and used "m" instead of "l") same result
Another result:

If: \[ y(n) = e^{j(\omega_0 n + \theta)} x(n) \]

Then: \[ r_{yy}[k] = e^{j\omega_0 k} r_{xx}[k] \]

\[ r_{yy}[k] = y[k] * y^*(-k) = y^*(-k) * y[k] \]

\[ = e^{-j(\omega_0 k + \theta)} x^*[-k] * e^{j(\omega_0 k + \theta)} x[k] \]

\[ = \sum_{h=-\infty}^{\infty} e^{j\omega_0 h} x^*[-h] e^{j\omega_0 (k-h)} x[(k-h)] \]

\[ = e^{j\omega_0 k} \sum_{h=-\infty}^{\infty} x^*(-h) x[(k-h)] = e^{j\omega_0 k} r_{xx}[k] \]