Sect. 2.6.2 Properties of Autocorrelation and Cross-Correlation Sequences

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\,x[n-\ell]$$

1. $|r_{xx}[\ell]| \leq r_{xx}[0]$ for all ℓ

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x = \text{energy}$$

2. $$\rho_{xx}[\ell] = \frac{r_{xx}[\ell]}{r_{xx}[0]}$$

$$|\rho_{xx}[\ell]| \leq 1 \quad \forall \ell$$

Proof: Tricky! (2)

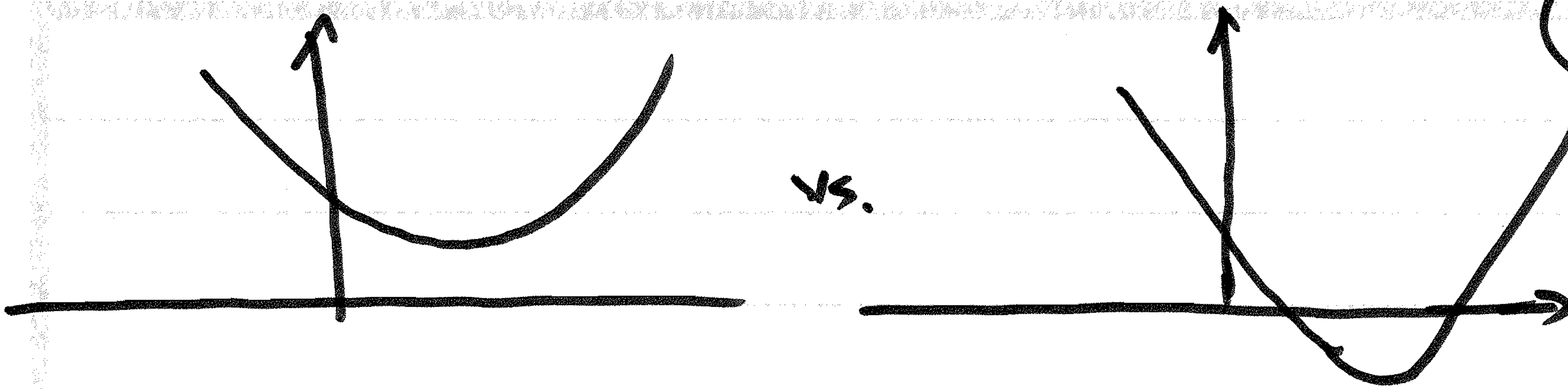
$$\sum_{n=-\infty}^{\infty} \{ z\, x[n] + y[n-\ell] \}^2 \geq 0$$

$$= z^2 \sum_{n=-\infty}^{\infty} x^2[n] + 2z \sum_{n=-\infty}^{\infty} x[n]\,y[n-\ell] + \sum_{n=-\infty}^{\infty} y^2[n-\ell]$$

$$= z^2 r_{xx}[0] + 2z\, r_{xy}[\ell] + r_{yy}[0] \geq 0$$

viewed as polynomial in z that can never go negative

$\Rightarrow$ must have complex-valued roots



Thus, discriminant must be such that $b^2 - 4ac < 0$

$$a z^2 + b z + c \quad \left\{ \begin{array}{l} a = r_{xx}[0] \\ b = 2\, r_{xy}[\ell] \\ c = r_{yy}[0] \end{array} \right.$$

THUS:

$$b^2 - 4ac = 4\, r_{xy}^2[\ell] - 4\, r_{xx}[0]\, r_{yy}[0] \le 0$$

$$r_{xy}^2[\ell] \le r_{xx}[0]\, r_{yy}[0]$$

$$|r_{xy}[\ell]| \le \sqrt{r_{xx}[0]\, r_{yy}[0]}$$

If $y[n] = x[n]$, then:

$$|r_{xx}[\ell]| \leq \sqrt{r_{xx}^2[0]}$$

$$|r_{xx}[\ell]| \leq r_{xx}[0]$$

Also, follows that with

$$\rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}}$$

then

$$|\rho_{xy}[\ell]| \leq 1 \quad \forall \ell$$

Summarizing three main properties of autocorrelation sequence $r_{xx}[\ell] = x[\ell] * x^*[-\ell]$

1. $r_{xx}[-\ell] = r_{xx}^*[\ell]$

2. $|r_{xx}[\ell]| \leq r_{xx}[0]$

3. $\sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell} \geq 0$

for all ω

From 1., $S_{xx}(\omega) = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell}$

is real-valued for all ω

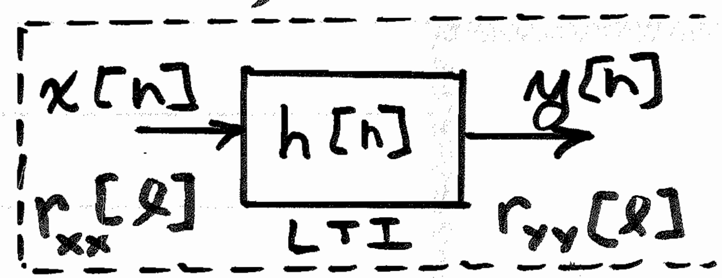
$S_{xx}(\omega) = X(\omega) X^*(\omega) = |X(\omega)|^2$

$\Rightarrow$ energy density spectrum

I/O Relationships for $r_{xx}[\ell]$, $r_{yx}[\ell]$, and $r_{yy}[\ell]$

Recall: $r_{xx}[\ell] = x[\ell] * x^*[-\ell]$

$$r_{yx}[\ell] = y[\ell] * x^*[-\ell]$$
$$= \left(h[\ell] * x[\ell]\right) * x^*[-\ell]$$
$$= h[\ell] * \left(x[\ell] * x^*[-\ell]\right)$$
$$= h[\ell] * r_{xx}[\ell]$$



Similarly:

$$r_{yy}[\ell] = r_{xx}[\ell] * r_{hh}[\ell]$$

Ideal Radar Problem Revisited

- <u>recall</u>: $x[n] * \delta[n-n_0] = x[n-n_0]$

- single target with no noise

$$y[n] = \Gamma\, x[n-D]$$

$$= x[n] * \underbrace{\Gamma\, \delta[n-D]}_{h[n] = \Gamma \delta[n-D]}$$

- can model as LTI system:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = \Gamma\, \delta[n-D]$$

We know: $r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$

- Thus: $r_{yx}[\ell] = \Gamma\, r_{xx}[\ell - D]$

Two-Target Case :

$$y[n] = T_1\, x[n-D_1] + T_2\, x[n-D_2]$$

$$= x[n] * \left\{ T_1\, \delta[n-D_1] + T_2\, \delta[n-D_2] \right\}$$

$$= x[n] * h[n]$$

Just a reminder: $x[n]$ is the transmitted signal, and we cross-correlate it with the received signal $y[n] \Rightarrow r_{yx}[\ell] = y[\ell] * x^*[-\ell]$ and look for peaks $\Rightarrow$ target round-trip delays

From previous result, we have:

$$r_{yx}[\ell] = r_{xx}[\ell] * h[\ell]$$

$$= T_1\, r_{xx}[\ell - D_1] + T_2\, r_{xx}[\ell - D_2]$$

Desire autocorrelation sequence that approximates a Kronecker Delta $\delta[n]$, especially to resolve closely-spaced targets

- else $r_{xx}[\ell - D_1]$ overlaps with $r_{xx}[\ell - D_2]$
- could yield only a single peak at some delay between D_1 and D_2
- or weaker (smaller) target masked by the "sidelobes" of the stronger target
- desire "constant modulus" signals, e.g. sequence of "+1"s and "-1"s, since for power amplifier efficiency, we operate in nonlinear region