Sect. 2.6 Autocorrelation / Cross-Correlation

- Motivation for Definition of Autocorrelation

- Detect if a certain signal is imbedded in a received signal (i.e., imbedded in some recorded time-series data set)
- In DT setting (sampled data), suppose you want to detect if the DT signal is imbedded in the recv'd data $y[n]$ and at what delay?
- Classic approach: run matched filter i.e., run data through filter matched to $x[n]$
- matched filter: $h[n] = x^*[-n]$
- run thru filter <u>means</u> convolve with impulse response $h[n]$

• Consider: $y[n] = a\,x[n-D] + \underbrace{w[n]}_{\text{e.g., noise}}$

• Convolve with matched filter

$h[n] = x^*[-n]$: $z[n] = y[n] * h[n]$

$= y[n] * x^*[-n]$

• Recall graphical approach to convolution, which involves time-reversing $h[n]$ and then sliding it along the input signal

• Since $h[n] = x^*[-n]$, the matched filter slides $x^*[n]$ along the input signal, sliding over one sample at a time, pt-wise multiplying and summing each time

$$y[n] * x^*[-n] = a\,x[n-D] * x^*[-n] + w[n] * x^*[-n]$$

$$= a\big(x[n] * \delta[n-D]\big) * x^*[-n] + w[n] * x^*[-n]$$

$$= x[n] * x^*[-n] * a\,\delta[n-D] + w[n] * x^*[-n]$$

$$= r_{xx}[n] * a\,\delta[n-D] + \underbrace{w[n] * x^*[-n]}_{\text{cross-correlation}}$$

$$= a\, r_{xx}[n-D] + r_{wx}[n]$$

• <u>where:</u> $r_{xx}[n] = x[n] * x^*[-n]$
is called auto-correlation of $x[n]$

and where we have invoked associativity and commutativity of convolution

- Ignore noise for the moment, but assume a second delayed replica of the signal $x[n]$ (corresponds to another target at a different range in radar app.)

$$y[n] = a_1 \, x[n-D_1] + a_2 \, x[n-D_2]$$

- Run matched-filter on $y[n]$:

$$y[n] * h[n] = y[n] * x^*[-n] =$$

$$= a_1 \left(x[n] * \delta[n-D_1] \right) * x^*[-n] + a_2 \left(x[n] * \delta[n-D_2] \right) * x^*[-n]$$

$$= x[n] * x^*[-n] * a_1 \delta[n-D_1] + \quad * x^*[-n] * a_2 \delta[n-D_2]$$

$$= a_1 \, r_{xx}[n-D_1] + a_2 \, r_{xx}[n-D_2]$$

- Obtain auto-correlation of $x[n]$ centered at each of the two delays, D_1 & D_2
- We desire to detect the presence of $x[n]$ in the data and determine the delays D_1 & D_2
- Thus, ideally desire $r_{xx}[n]$ to be as close as possible to a Delta Function
- Desire $r_{xx}(n)$ to be "spiky" and tower above the noisy background
- Under practical constraints, e.g., nonlinear amplifiers

- Examine: auto-correlation

$$r_{xx}[n] = x[n] * x^*[-n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, x^*[-(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, x^*[k-n]$$

- <u>NOW</u> historically, researchers define a time-difference of arrival lag variable ℓ and let the summation over k above run over DT time variable n (rather than k)

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-\ell]$$

- So, then we express auto-correlation as:

$$r_{xx}[\ell] = x[\ell] * x^*[-\ell]$$

- Since the textbook somes m for the lag variabe in the stochastic version of auto-correlation and cross-correlation later in the book, I sometimes use ℓ and m interchangeably when you look at old exam problems

e.g. $$r_{xx}[m] = x[m] * x^*[-m]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-m]$$