10.3.4 Characteristics of Commonly Used Analog Filters

As we have seen from our discussion above, IIR digital filters can easily be obtained by beginning with an analog filter and then using a mapping to transform the s-plane into the z-plane. Thus the design of a digital filter is reduced to designing an appropriate analog filter and then performing the conversion from $H(s)$ to $H(z)$, in such a way so as to preserve, as much as possible, the desired characteristics of the analog filter.

Analog filter design is a well-developed field and many books have been written on the subject. In this section we briefly describe the important characteristics of commonly used analog filters and introduce the relevant filter parameters. Our discussion is limited to lowpass filters. Subsequently, we describe several frequency transformations that convert a lowpass prototype filter into either a bandpass, highpass, or band-elimination filter.

Butterworth filters. Lowpass Butterworth filters are all-pole filters characterized by the magnitude-squared frequency response

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}} \tag{10.3.45}$$

where N is the order of the filter, Ω_c is its -3-dB frequency (usually called the cutoff frequency), Ω_p is the passband edge frequency, and $1/(1+\epsilon^2)$ is the band-edge value of $|H(\Omega)|^2$. Since $H(s)H(-s)$ evaluated at $s = j\Omega$ is simply equal to $|H(\Omega)|^2$, it follows that

$$H(s)H(-s) = \frac{1}{1+(-s^2/\Omega_c^2)^N} \tag{10.3.46}$$

The poles of $H(s)H(-s)$ occur on a circle of radius Ω_c at equally spaced points. From (10.3.46) we find that

$$\frac{-s^2}{\Omega_c^2} = (-1)^{1/N} = e^{j(2k+1)\pi/N}, \qquad k = 0, 1, \ldots, N-1$$

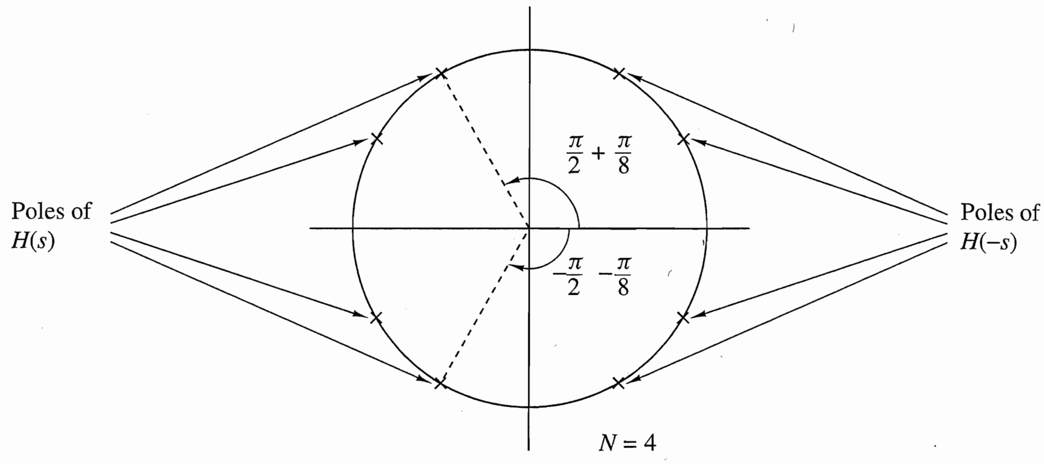
and hence

$$s_k = \Omega_c e^{j\pi/2} e^{j(2k+1)\pi/2N}, \qquad k = 0, 1, \ldots, N-1 \tag{10.3.47}$$

For example, Fig. 10.3.9 illustrates the pole positions for $N = 4$ and $N = 5$ Butterworth filters.

The frequency response characteristics of the class of Butterworth filters are shown in Fig. 10.3.10 for several values of N. We note that $|H(\Omega)|^2$ is monotonic in both the passband and stopband. The order of the filter required to meet an attenuation δ_2 at a specified frequency Ω_s is easily determined from (10.3.45). Thus at $\Omega = \Omega_s$ we have

$$\frac{1}{1+\epsilon^2(\Omega_s/\Omega_p)^{2N}} = \delta_2^2$$



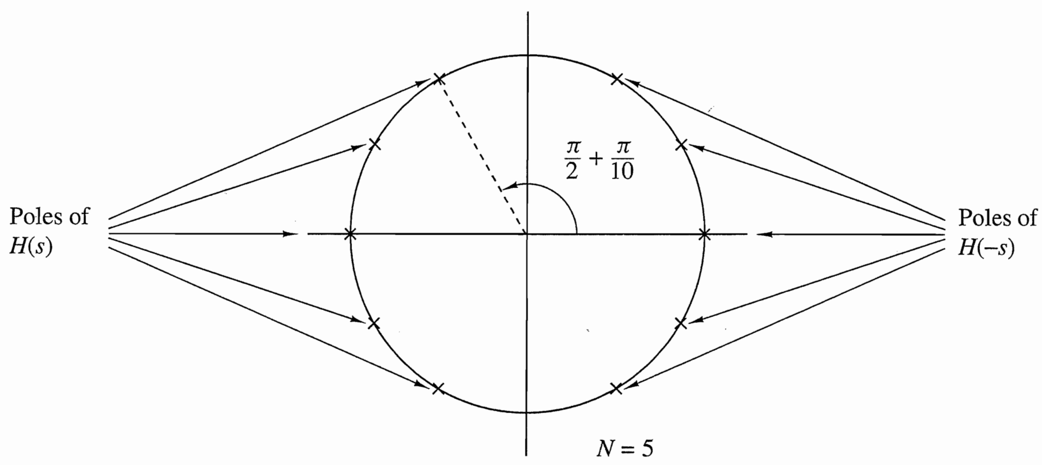


Figure 10.3.9 Pole positions for Butterworth filters.

and hence

$$N = \frac{\log[(1/\delta_2^2) - 1]}{2\log(\Omega_s/\Omega_c)} = \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)} \tag{10.3.48}$$

where, by definition, $\delta_2 = 1/\sqrt{1+\delta^2}$. Thus the Butterworth filter is completely characterized by the parameters N, δ_2, ϵ, and the ratio Ω_s/Ω_p.

EXAMPLE 10.3.6

Determine the order and the poles of a lowpass Butterworth filter that has a −3-dB bandwidth of 500 Hz and an attenuation of 40 dB at 1000 Hz.

Solution. The critical frequencies are the −3-dB frequency Ω_c and the stopband frequency Ω_s, which are

$$\Omega_c = 1000\pi$$

$$\Omega_s = 2000\pi$$

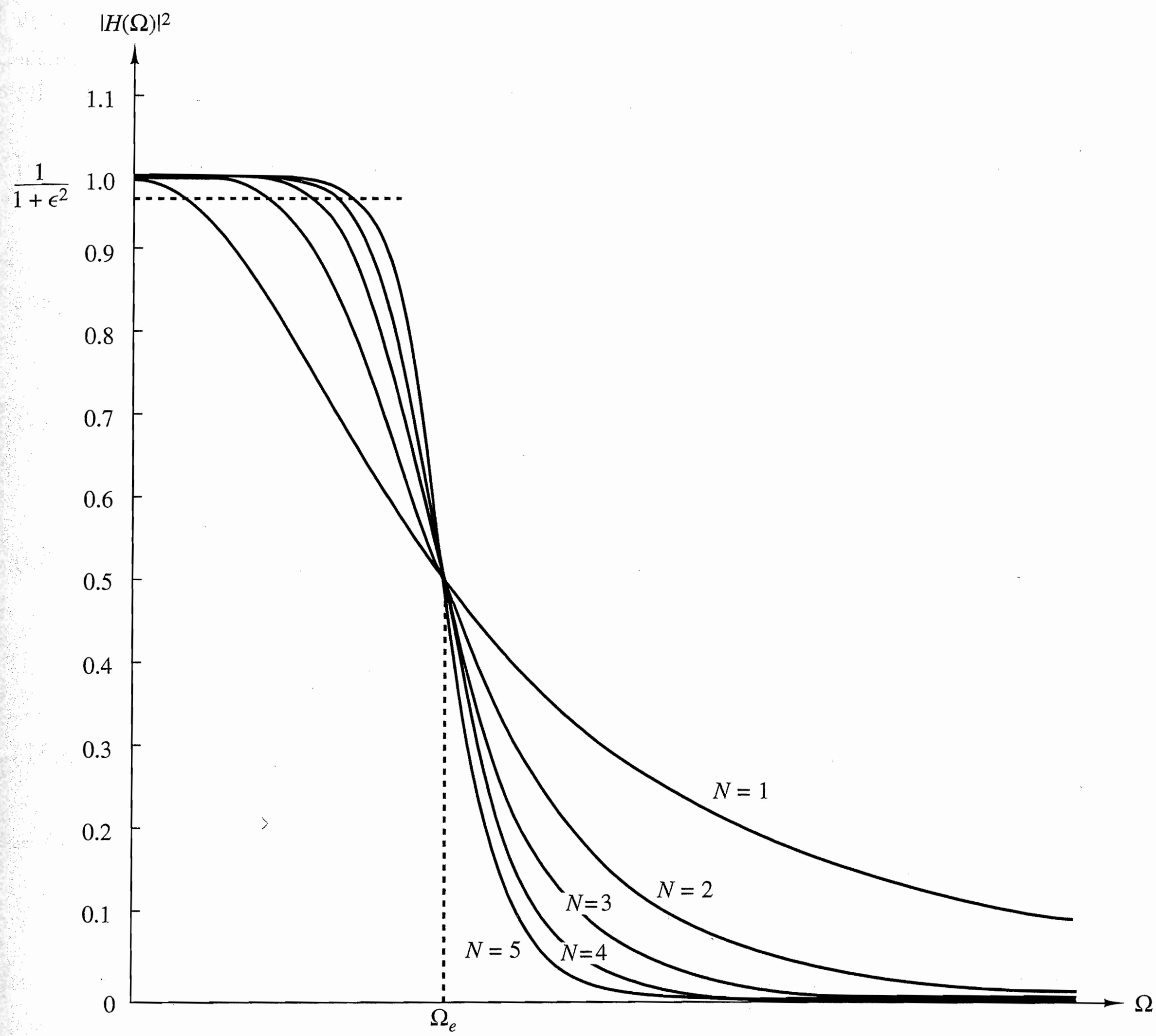


Figure 10.3.10 Frequency response of Butterworth filters.

For an attenuation of 40 dB, $\delta_2 = 0.01$. Hence from (10.3.48) we obtain

$$N = \frac{\log_{10}(10^4 - 1)}{2\log_{10} 2}$$

$$= 6.64$$

To meet the desired specifications, we select $N = 7$. The pole positions are

$$s_k = 1000\pi e^{j[\pi/2+(2k+1)\pi/14]}, \qquad k = 0, 1, 2, \ldots, 6$$

Chebyshev filters. There are two types of Chebyshev filters. Type I Chebyshev filters are all-pole filters that exhibit equiripple behavior in the passband and a monotonic characteristic in the stopband. On the other hand, the family of type II Chebyshev filters contains both poles and zeros and exhibits a monotonic behavior in the passband and an equiripple behavior in the stopband. The zeros of this class of filters lie on the imaginary axis in the s-plane.

The magnitude squared of the frequency response characteristic of a type I Chebyshev filter is given as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)} \tag{10.3.49}$$

where ϵ is a parameter of the filter related to the ripple in the passband and $T_N(x)$ is the Nth-order Chebyshev polynomial defined as

$$T_N(x) = \begin{cases} \cos(N\cos^{-1} x), & |x| \le 1 \\ \cosh(N\cosh^{-1} x), & |x| > 1 \end{cases} \tag{10.3.50}$$

The Chebyshev polynomials can be generated by the recursive equation

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x), \qquad N = 1, 2, \ldots \tag{10.3.51}$$

where $T_0(x) = 1$ and $T_1(x) = x$. From (10.3.51) we obtain $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, and so on.

Some of the properties of these polynomials are as follows:

1. $|T_N(x)| \le 1$ for all $|x| \le 1$.
2. $T_N(1) = 1$ for all N.
3. All the roots of the polynomial $T_N(x)$ occur in the interval $-1 \le x \le 1$.

The filter parameter ϵ is related to the ripple in the passband, as illustrated in Fig. 10.3.11, for N odd and N even. For N odd, $T_N(0) = 0$ and hence $|H(0)|^2 = 1$. On the other hand, for N even, $T_N(0) = 1$ and hence $|H(0)|^2 = 1/(1+\epsilon^2)$. At the band-edge frequency $\Omega = \Omega_p$, we have $T_N(1) = 1$, so that

$$\frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1$$

or, equivalently,

$$\epsilon^2 = \frac{1}{(1-\delta_1)^2} - 1 \tag{10.3.52}$$

where δ_1 is the value of the passband ripple.

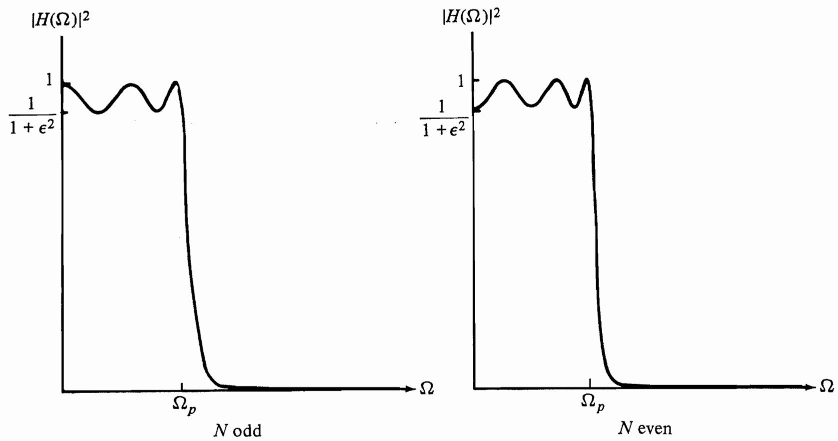


Figure 10.3.11 Type I Chebyshev filter characteristic.

The poles of a type I Chebyshev filter lie on an ellipse in the s-plane with major axis

$$r_1 = \Omega_p \frac{\beta^2 + 1}{2\beta} \tag{10.3.53}$$

and minor axis

$$r_2 = \Omega_p \frac{\beta^2 - 1}{2\beta} \tag{10.3.54}$$

where β is related to ϵ according to the equation

$$\beta = \left[\frac{\sqrt{1+\epsilon^2} + 1}{\epsilon} \right]^{1/N} \tag{10.3.55}$$

The pole locations are most easily determined for a filter of order N by first locating the poles for an equivalent Nth-order Butterworth filter that lie on circles of radius r_1 or radius r_2, as illustrated in Fig. 10.3.12. If we denote the angular positions of the poles of the Butterworth filter as

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0, 1, 2, \ldots, N-1 \tag{10.3.56}$$

then the positions of the poles for the Chebyshev filter lie on the ellipse at the coordinates (x_k, y_k), $k = 0, 1, \ldots, N-1$, where

$$\begin{aligned} x_k &= r_2 \cos \phi_k, \quad k = 0, 1, \ldots, N-1 \\ y_k &= r_1 \sin \phi_k, \quad k = 0, 1, \ldots, N-1 \end{aligned} \tag{10.3.57}$$

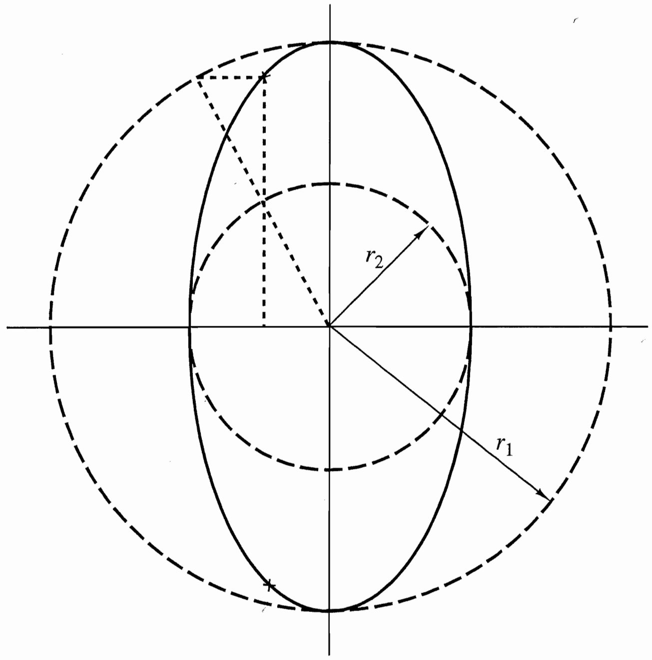


Figure 10.3.12
Determination of the pole locations for a Chebyshev filter.

A type II Chebyshev filter contains zeros as well as poles. The magnitude squared of its frequency response is given as

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2[T_N^2(\Omega_s/\Omega_p)/T_N^2(\Omega_s/\Omega)]} \tag{10.3.58}$$

where $T_N(x)$ is, again, the Nth-order Chebyshev polynomial and Ω_s is the stopband frequency as illustrated in Fig. 10.3.13. The zeros are located on the imaginary axis at the points

$$s_k = j\frac{\Omega_s}{\sin\phi_k}, \qquad k = 0, 1, \ldots, N-1 \tag{10.3.59}$$

The poles are located at the points (v_k, w_k), where

$$v_k = \frac{\Omega_s x_k}{\sqrt{x_k^2 + y_k^2}}, \qquad k = 0, 1, \ldots, N-1 \tag{10.3.60}$$

$$w_k = \frac{\Omega_s y_k}{\sqrt{x_k^2 + y_k^2}}, \qquad k = 0, 1, \ldots, N-1 \tag{10.3.61}$$

where $\{x_k\}$ and $\{y_k\}$ are defined in (10.3.57) with β now related to the ripple in the stopband through the equation

$$\beta = \left[\frac{1+\sqrt{1-\delta_2^2}}{\delta_2}\right]^{1/N} \tag{10.3.62}$$

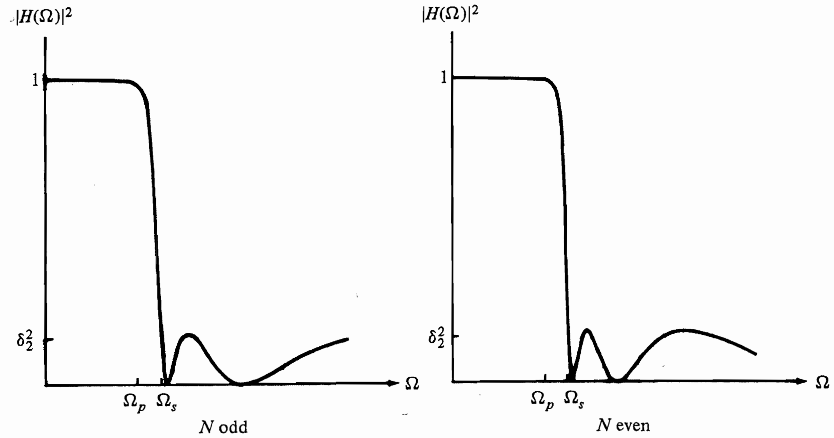


Figure 10.3.13 Type II Chebyshev filters.

From this description, we observe that the Chebyshev filters are characterized by the parameters N, ϵ, δ_2, and the ratio Ω_s/Ω_p. For a given set of specifications on ϵ, δ_2, and Ω_s/Ω_p, we can determine the order of the filter from the equation

$$N = \frac{\log\left[\left(\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\epsilon^2)}\right)/\epsilon\delta_2\right]}{\log\left[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}\right]} \tag{10.3.63}$$

$$= \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

where, by definition, $\delta_2 = 1/\sqrt{1+\delta^2}$.

EXAMPLE 10.3.7

Determine the order and the poles of a type I lowpass Chebyshev filter that has a 1-dB ripple in the passband, a cutoff frequency $\Omega_p = 1000\pi$, a stopband frequency of 2000π, and an attenuation of 40 dB or more for $\Omega \geq \Omega_s$.

Solution. First, we determine the order of the filter. We have

$$10\log_{10}(1+\epsilon^2) = 1$$

$$1+\epsilon^2 = 1.259$$

$$\epsilon^2 = 0.259$$

$$\epsilon = 0.5088$$

Also,

$$20\log_{10}\delta_2 = -40$$

$$\delta_2 = 0.01$$

Hence from (10.3.63) we obtain

$$N = \frac{\log_{10} 196.54}{\log_{10}(2+\sqrt{3})}$$
$$= 4.0$$

Thus a type I Chebyshev filter having four poles meets the specifications.

The pole positions are determined from the relations in (10.3.53) through (10.3.57). First, we compute β, r_1, and r_2. Hence

$$\beta = 1.429$$
$$r_1 = 1.06\Omega_p$$
$$r_2 = 0.365\Omega_p$$

The angles $\{\phi_k\}$ are

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{8}, \qquad k = 0, 1, 2, 3$$

Therefore, the poles are located at

$$x_1 + jy_1 = -0.1397\Omega_p \pm j0.979\Omega_p$$
$$x_2 + jy_2 = -0.337\Omega_p \pm j0.4056\Omega_p$$

The filter specifications in Example 10.3.7 are very similar to the specifications given in Example 10.3.6, which involved the design of a Butterworth filter. In that case the number of poles required to meet the specifications was seven. On the other hand, the Chebyshev filter required only four. This result is typical of such comparisons. In general, the Chebyshev filter meets the specifications with fewer poles than the corresponding Butterworth filter. Alternatively, if we compare a Butterworth filter to a Chebyshev filter having the same number of poles and the same passband and stopband specifications, the Chebyshev filter will have a smaller transition bandwidth. For a tabulation of the characteristics of Chebyshev filters and their pole–zero locations, the interested reader is referred to the handbook of Zverev (1967).

Elliptic filters. Elliptic (or Cauer) filters exhibit equiripple behavior in both the passband and the stopband, as illustrated in Fig. 10.3.14 for N odd and N even. This class of filters contains both poles and zeros and is characterized by the magnitude-squared frequency response

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 U_N(\Omega/\Omega_p)} \tag{10.3.64}$$

where $U_N(x)$ is the Jacobian elliptic function of order N, which has been tabulated by Zverev (1967), and ϵ is a parameter related to the passband ripple. The zeros lie on the $j\Omega$-axis.

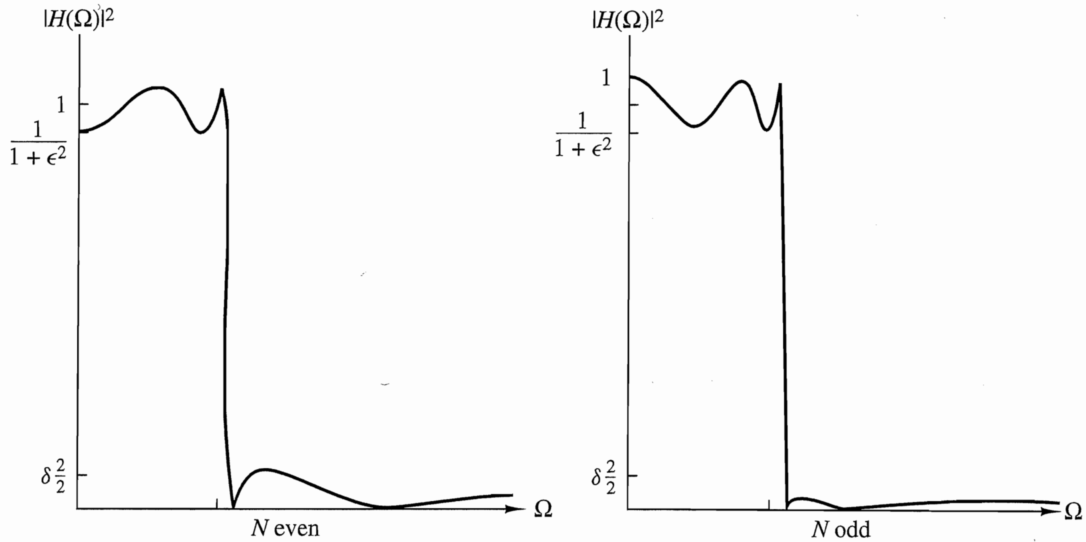


Figure 10.3.14 Magnitude-squared frequency characteristics of elliptic filters.

We recall from our discussion of FIR filters that the most efficient designs occur when we spread the approximation error equally over the passband and the stopband. Elliptic filters accomplish this objective and, as a consequence, are the most efficient from the viewpoint of yielding the smallest-order filter for a given set of specifications. Equivalently, we can say that for a given order and a given set of specifications, an elliptic filter has the smallest transition bandwidth.

The filter order required to achieve a given set of specifications in passband ripple δ_1, stopband ripple δ_2, and transition ratio Ω_p/Ω_s is given as

$$N = \frac{K(\Omega_p/\Omega_s)K\left(\sqrt{1-(\epsilon^2/\delta^2)}\right)}{K(\epsilon/\delta)K\left(\sqrt{1-(\Omega_p/\Omega_s)^2}\right)} \tag{10.3.65}$$

where $K(x)$ is the complete elliptic integral of the first kind, defined as

$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} \tag{10.3.66}$$

and $\delta_2 = 1/\sqrt{1+\delta^2}$. Values of this integral have been tabulated in a number of texts [e.g., the books by Jahnke and Emde (1945) and Dwight (1957)]. The passband ripple is $10\log_{10}(1+\epsilon^2)$.

We shall not attempt to describe elliptic functions in any detail because such a discussion would take us too far afield. Suffice to say that computer programs are available for designing elliptic filters from the frequency specifications indicated above.

In view of the optimality of elliptic filters, the reader may question the reason for considering the class of Butterworth or the class of Chebyshev filters in practical applications. One important reason that these other types of filters might be preferable in some applications is that they possess better phase response characteristics. The phase response of elliptic filters is more nonlinear in the passband than a comparable Butterworth filter or a Chebyshev filter, especially near the band edge.

This filter has poles at

$$p_{1,2} = 0.987e^{\pm j\pi/2}$$

and zeros at

$$z_{1,2} = -1, 0.95$$

Therefore, we have succeeded in designing a two-pole filter that resonates near $\omega = \pi/2$.

In this example the parameter T was selected to map the resonant frequency of the analog filter into the desired resonant frequency of the digital filter. Usually, the design of the digital filter begins with specifications in the digital domain, which involve the frequency variable ω. These specifications in frequency are converted to the analog domain by means of the relation in (10.3.43). The analog filter is then designed that meets these specifications and converted to a digital filter by means of the bilinear transformation in (10.3.40). In this procedure, the parameter T is transparent and may be set to any arbitrary value (e.g., $T = 1$). The following example illustrates this point.

EXAMPLE 10.3.5

Design a single-pole lowpass digital filter with a 3-dB bandwidth of 0.2π, using the bilinear transformation applied to the analog filter

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

where Ω_c is the 3-dB bandwidth of the analog filter.

Solution. The digital filter is specified to have its -3-dB gain at $\omega_c = 0.2\pi$. In the frequency domain of the analog filter $\omega_c = 0.2\pi$ corresponds to

$$\Omega_c = \frac{2}{T}\tan 0.1\pi$$

$$= \frac{0.65}{T}$$

Thus the analog filter has the system function

$$H(s) = \frac{0.65/T}{s + 0.65/T}$$

This represents our filter design in the analog domain.

Now, we apply the bilinear transformation given by (10.3.40) to convert the analog filter into the desired digital filter. Thus we obtain

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

where the parameter T has been divided out.

The frequency response of the digital filter is

$$H(\omega) = \frac{0.245(1 + e^{-j\omega})}{1 - 0.509e^{-j\omega}}$$

At $\omega = 0$, $H(0) = 1$, and at $\omega = 0.2\pi$, we have $|H(0.2\pi)| = 0.707$, which is the desired response.
