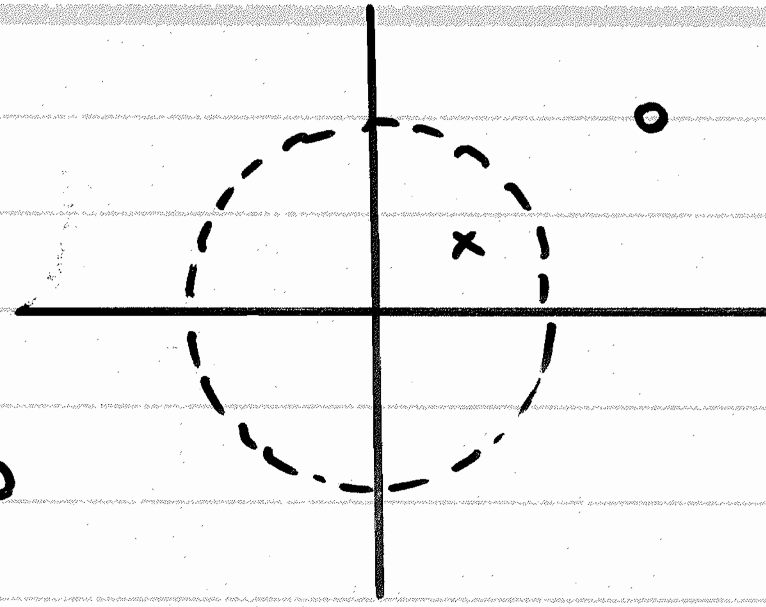
Sect. 5.4.6 Notch Filters

Note wrt All-Pass Filters.

- mathematical preliminary
- let $c = a + jb = |c| e^{j\angle c}$
- note: $\dfrac{c}{c^*} = \dfrac{|c| e^{j\angle c}}{|c| e^{-j\angle c}} = 1\, e^{j2\angle c}$
- THUS: $\left|\dfrac{c}{c^*}\right| = 1$
- Now, consider system with single pole at $z = p$ and a single zero at $z = \dfrac{1}{p^*}$

$$H(z) = G\,\frac{\left(z - \frac{1}{p^*}\right)}{z - p}$$



$$H(\omega) = H(z)\Big|_{z = e^{j\omega}}$$

$$H(\omega) = G\,\frac{\left(e^{j\omega} - \frac{1}{p^*}\right)}{e^{j\omega} - p} = \frac{G\,\frac{1}{p^*}\,e^{+j\omega}\left(p^* - e^{-j\omega}\right)}{-\left(p - e^{j\omega}\right)}$$

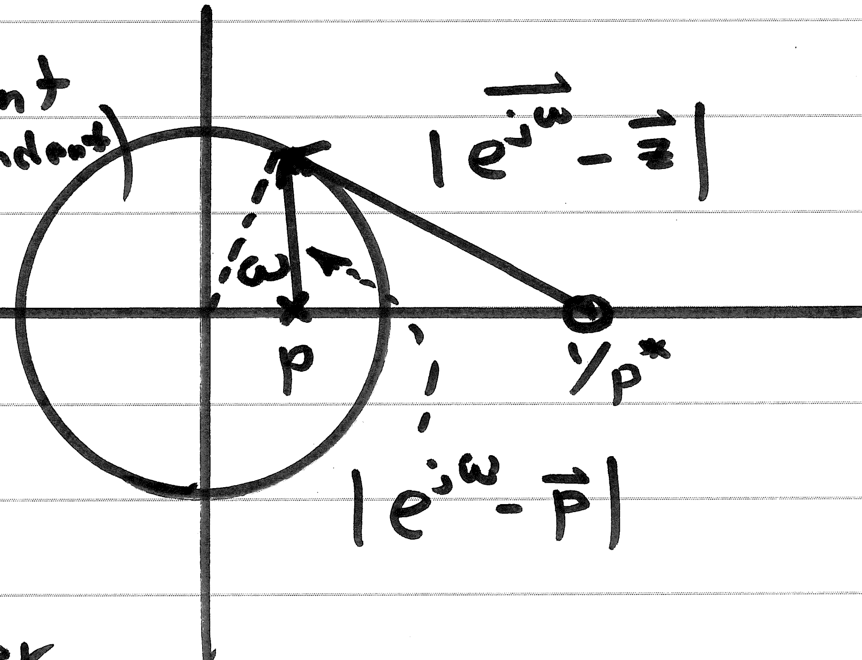
$$= -\frac{G}{p^*}\,e^{j\omega}\,\frac{c}{c^*} \qquad \text{where: } c = p - e^{j\omega}$$

THUS: $|H(\omega)| = \left|\frac{G}{p^*}\right| = \frac{|G|}{|p|}$ } does not depend on ω

$\Rightarrow$ ALL PASS!

All-Pass Filter

$$\frac{|\overrightarrow{e^{j\omega}} - \vec{z}|}{|\overrightarrow{e^{j\omega}} - \vec{p}|} = \text{constant} \quad \left(\text{independent of } \omega\right)$$



For all-pass filter, ratio of the two "lengths" stays same as you vary ω

$$\frac{|\cos\omega + j\sin\omega - \frac{1}{p}|^2}{|\cos\omega + j\sin\omega - p|^2} = \frac{(\cos\omega - \frac{1}{p})^2 + \sin^2\omega}{(\cos\omega - p)^2 + \sin^2\omega} = \frac{1 - \frac{2}{p}\cos\omega + \frac{1}{p^2}}{1 - 2p\cos\omega + p^2}$$

$$= \frac{1}{p^2}\,\frac{(p^2 - 2p\cos\omega + 1)}{(p^2 - 2p\cos\omega + 1)} = \frac{1}{p^2} = |H(\omega)|^2$$

- An all-pass filter can be used to stabilize an unstable system without affecting the magnitude of the frequency response
- Suppose there is a pole at p outside unit circle $\Rightarrow$ $1/p^*$ is inside unit circle

$$\underbrace{\frac{H'(z)}{(z-p)}}_{\text{everything but } (z-p)} \times \frac{(z-p)}{(z-1/p^*)}$$

zero-pole cancellation

now have new pole at $z = 1/p^*$ inside unit circle

$\Rightarrow$ magnitude is unaffected

- Consider real-valued all-pass filter with single-pole:

$$H(z) = p\,\frac{z - 1/p}{z - p} \Rightarrow |H(\omega)| = 1 \quad \forall\, \omega$$

$$r_{hh}[\ell] = h[\ell] * h^*[-\ell] \xleftrightarrow{\text{DTFT}} |H(\omega)|^2 = 1$$

$$= \delta[\ell]$$

- What is $h[n]$?

$$H(z) = p\,\frac{z}{z-p} - z^{-1}\frac{z}{z-p}$$

- Thus:

$$h[n] = p\;p^n u[n] - p^{n-1} u[n-1]$$

$$= p\;p^n u[n] - \frac{1}{p}\,p^n\,u[n-1]$$

- Some algebraic manipulation:

$$h[n] = \alpha\,\delta[n] + \left(p - \frac{1}{p}\right) p^n u[n]$$

- where: $p = \alpha + p - \frac{1}{p} \Rightarrow \alpha = \frac{1}{p}$

$$h[n] = \frac{1}{p}\,\delta[n] + \left(\frac{p^2-1}{p}\right) p^n u[n]$$

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let: $x[n] = \frac{1}{p} \left\{ \delta[n] + (p^2-1)\, p^n u[n] \right\}$

- One can verify that:

$$r_{xx}[\ell] = x[\ell] * x[-\ell] = \delta[\ell]$$

- note: $x[n]$ is <u>not</u> constant modulus

- also, recall:

$$x[n] = p^n u[n] \Rightarrow r_{xx}[\ell] = \frac{1}{1-p^2}\, p^{|\ell|}$$