I. Introduction

1. The biquadratic transfer function has both a 2nd order numerator and a 2nd order denominator:

\[ H(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \]

2. Review: Basic Inverting Amplifier: \( V_{out} = -\frac{Z_f(s)}{Z_{in}(s)} V_{in} = -\frac{Y_{in}(s)}{Y_f(s)} V_{in} \)

3. Basic Inverting Summer: \( v_{out} = -G_1 v_1 - G_2 v_2 - G_3 v_3 \)
4. Basic Inverting Integrator: \( v_{out} = -G_1 \int v_1 - G_2 \int v_2 - G_3 \int v_3 \)
II. Controllable Canonical 4 OP AMP State Space Realization of the Biquadratic TF: A CONCEPTUAL Block Diagram Development

1. Recall again the biquadratic structure with \( a_1 > 0 \) and \( a_2 > 0 \) to insure stability (necessary and sufficient conditions for poles in the left half complex plane):

\[
H(s) = \frac{V_{out}}{V_{in}} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} = \frac{b_0 s^2 + b_1 s + b_2}{d(s)}
\]

\[
= \frac{b_2}{d(s)} + \frac{b_1 s}{d(s)} + \frac{b_0 s^2}{d(s)}
\]

where \( d(s) = s^2 + a_1 s + a_2 \). Since \( V_{out}(s) = H(s)V_{in}(s) \),

\[
V_{out}(s) = b_2 \left( \frac{V_{in}(s)}{d(s)} \right) + b_1 s \left( \frac{V_{in}(s)}{d(s)} \right) + b_0 s^2 \left( \frac{V_{in}(s)}{d(s)} \right)
\]

2. Define the AUXILIARY EQUATION:

\[
\tilde{V}_{out}(s) = \frac{1}{d(s)} V_{in}(s) = \frac{1}{s^2 + a_1 s + a_2} V_{in}(s)
\]

(a) Implication 1: s-domain auxiliary equation

\[
(s^2 + a_1 s + a_2)\tilde{V}_{out}(s) = V_{in}(s)
\]

(b) Implication 2: s-domain output equation

\[
V_{out}(s) = b_2 \frac{V_{in}(s)}{d(s)} + b_1 s \frac{V_{in}(s)}{d(s)} + b_0 s^2 \frac{V_{in}}{d(s)}
\]

\[
= b_2 \tilde{V}_{out} + b_1 s \tilde{V}_{out} + b_0 s^2 \tilde{V}_{out}
\]
3. **OBJECTIVE**: develop a flow diagram that describes $\hat{V}_{out}(s)$ and then enhance this flow diagram to obtain $V_{out}(s)$. There are two critical s-domain equations here:

(a) *(auxiliary equation s-domain)*

\[
(s^2 + a_1s + a_2)\hat{V}_{out}(s) = s^2\hat{V}_{out}(s) + a_1s\hat{V}_{out}(s) + a_2\hat{V}_{out}(s) = V_{in}(s)
\]

and

(b) *(output equation s-domain)*

\[
V_{out} = b_2\hat{V}_{out} + b_1s\hat{V}_{out} + b_0s^2\hat{V}_{out}
\]

4. **Interpret s-domain equations as differential equations in the t-world**. Recall, multiplication by $s$ means differentiation in the time world. (The number of dots over the time variables means the order of the time derivative.)

(a) $s^2\hat{V}_{out}(s) + a_1s\hat{V}_{out}(s) + a_2\hat{V}_{out}(s) = V_{in}(s)$ implies that in the t-world:

\[
\ddot{v}_{out}(t) + a_1\dot{v}_{out}(t) + a_2v_{out}(t) = v_{in}(t)
\]

or equivalently

\[
\ddot{v}_{out}(t) = v_{in}(t) - a_1\dot{v}_{out}(t) - a_2v_{out}(t)
\]

(b) $V_{out} = b_2\hat{V}_{out} + b_1s\hat{V}_{out} + b_0s^2\hat{V}_{out}$ implies that in the t-world:

\[
v_{out}(t) = b_2\hat{v}_{out}(t) + b_1\dot{v}_{out}(t) + b_0\ddot{v}_{out}(t)
\]

(c) **Conclusion**: $\ddot{v}_{out}(t) = v_{in}(t) - a_1\dot{v}_{out}(t) - a_2v_{out}(t)$ implies
\[
v_{out} = (b_2 - b_0a_2)\ddot{v}_{out} + (b_1 - b_0a_1)\dot{v}_{out} + b_0v_{in}
\]

5. **Summary**: Two important relationships result:

   (a) **Auxiliary Equation in t-world**:
   \[
   \ddot{v}_{out}(t) = v_{in}(t) - a_1\dot{v}_{out}(t) - a_2v_{out}(t)
   \]

   (b) **Output Equation in t-world**:
   \[
   v_{out} = (b_2 - b_0a_2)\ddot{v}_{out} + (b_1 - b_0a_1)\dot{v}_{out} + b_0v_{in}
   \]

6. **Block Diagram Interpretation of auxiliary equation** since we do not know the signs on the coefficients and hence cannot generate the actual op amp circuit:

   ![Block Diagram](image)

**Example 1**: Realize \( H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{0.1s^2 + 0.3s + 0.7}{s^2 + 2s + 4} \) using the controllable canonical realization.

**Step 1.** Time domain auxiliary equation is:
\[ \ddot{v}_{\text{out}}(t) = v_{\text{in}}(t) - 2\dot{v}_{\text{out}}(t) - 4\ddot{v}_{\text{out}}(t) \]

**Step 2.** Integrate both sides and change signs to obtain:

\[
-\ddot{v}_{\text{out}}(t) = -\int_{-\infty}^{t} \ddot{v}_{\text{out}}(\tau) d\tau \\
= -\int_{-\infty}^{t} v_{\text{in}}(\tau) d\tau - 2\int_{-\infty}^{t} [\ddot{v}_{\text{out}}(\tau)] d\tau - 4\int_{-\infty}^{t} [\dddot{v}_{\text{out}}(\tau)] d\tau
\]

**Step 3.** Realize equation of step 2:

**Step 4.** Recall \( v_{\text{out}}(t) = 0.7\ddot{v}_{\text{out}}(t) + 0.3\dot{v}_{\text{out}}(t) + 0.1\dddot{v}_{\text{out}}(t) \) and

\[ \ddot{v}_{\text{out}}(t) = v_{\text{in}}(t) - 2\dot{v}_{\text{out}}(t) - 4\ddot{v}_{\text{out}}(t) \] in which case
\[-v_{\text{out}}(t) = -0.1v_{\text{in}}(t) - 0.3\dot{v}_{\text{out}}(t) - 0.1\ddot{v}_{\text{out}}(t)\]

which has realization:

![Diagram](image)

**Part 2: The Observable Canonical Form—by example only**

**Example 2:** Realize \(H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{0.1s^2 + 0.3s + 0.7}{s^2 + 2s + 4}\) using the observable canonical realization.

**Step 1.** Given a transfer function, generate a differential equation in the input and output.

\[H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{0.1s^2 + 0.3s + 0.7}{s^2 + 2s + 4}\]

implies

\[(s^2 + 2s + 4)V_{\text{out}}(s) = (0.1s^2 + 0.3s + 0.7)V_{\text{in}}(s)\]

implies
\[ \dot{v}_{out}(t) + 2\ddot{v}_{out}(t) + 4v_{out}(t) = 0.7v_{in}(t) + 0.3\dot{v}_{in}(t) + 0.1\ddot{v}_{in}(t) \]

**Step 2.** Replace derivative by the D-operator in which \( D^k = \frac{d^k}{dt^k} \) and \( D^{-k} \) is the k-fold integral.

\[ D^2v_{out}(t) + 2Dv_{out}(t) + 4v_{out}(t) = 0.7v_{in}(t) + 0.3Dv_{in}(t) + 0.1Dv_{in}(t) \]

Equivalently

\[ D^2v_{out}(t) = 0.1D^2v_{in}(t) + D\left[0.3v_{in}(t) - 2v_{out}(t)\right] + D^{-2}\left[0.7v_{in}(t) - 4v_{out}(t)\right] \]

Multiplying both sides by \( D^{-2} \) yields what one would call an integral equation.

\[ v_{out}(t) = 0.1v_{in}(t) + D^{-1}\left[0.3v_{in}(t) - 2v_{out}(t)\right] + D^{-2}\left[0.7v_{in}(t) - 4v_{out}(t)\right] \]

**Step 3.** Define an intermediate variable \( x_1 \) and realize resulting equation as an op amp circuit assuming \( x_1 \) is available—it will be made available in step 4. In general the intermediate variable has no physical meaning.

(a) From step 2, recall that

\[ v_{out}(t) = 0.1v_{in}(t) + D^{-1}\left[0.3v_{in}(t) - 2v_{out}(t)\right] + D^{-2}\left[0.7v_{in}(t) - 4v_{out}(t)\right] \]

which has a bunch of messy implicit integrals.

(b) Define an intermediate variable

\[ x_1 = D^{-1}\left[0.3v_{in}(t) - 2v_{out}(t)\right] + D^{-2}\left[0.7v_{in}(t) - 4v_{out}(t)\right] \]
to “cover up” the implicit integrals.

(c) Thus

\[ v_{out}(t) = 0.1v_{in}(t) + x_1 \quad \text{or} \quad -v_{out}(t) = -0.1v_{in}(t) - x_1 \]

(d) Op amp realization assuming \( x_1 \) is available:

![Op amp circuit diagram]

Step 4. Construct a differential equation for \( x_1 \).

(a) Recall

\[ x_1 = D^{-1}[0.3v_{in}(t) - 2v_{out}(t)] + D^{-2}[0.7v_{in}(t) - 4v_{out}(t)] \]

(b) Multiply both sides by \( D \) to obtain

\[ \dot{x}_1 = Dx_1 = 0.3v_{in}(t) - 2v_{out}(t) + D^{-1}[0.7v_{in}(t) - 4v_{out}(t)] \]

(c) To make life simpler by making the implicit integrals disappear define a second intermediate variable \( x_2 \):

\[ x_2 = D^{-1}[0.7v_{in}(t) - 4v_{out}(t)] \]

(d) Thus

\[ \dot{x}_1 = Dx_1 = 0.3v_{in}(t) - 2v_{out}(t) + x_2 \]

(e) Substitute for \( v_{out}(t) \) where from step 3 \( v_{out}(t) = 0.1v_{in}(t) + x_1 \).
(f) It follows that \( \dot{x}_1 = 0.1v_{in}(t) - 2x_1 + x_2 \).

**Step 5.** Realization of (f) in step 4.

(a) Integrating both sides of \( \dot{x}_1 = 0.1v_{in}(t) - 2x_1 + x_2 \) and multiplying by “-1” yields:

\[
-x_1(t) = - \int_{-\infty}^{t} \dot{x}_1(\tau) d\tau = -0.1 \int_{-\infty}^{t} v_{in}(\tau) d\tau - 2 \int_{-\infty}^{t} [-x_1(\tau)] d\tau - \int_{-\infty}^{t} x_2(\tau) d\tau
\]

(b) This integral equation can be realized by the op amp integrator circuit below:

![Op Amp Integrator Circuit Diagram]

Notice that an additional op amp is used to drive the op amp circuit of step 3 which requires \( x_1 \).

**Step 6.** Build an op amp circuit to generate \( x_2 \). Consider that

\[
x_2 = D^{-1}[0.7v_{in}(t) - 4v_{out}(t)]
\]

implies that

\[
\dot{x}_2(t) = Dx_2(t) = 0.7v_{in}(t) - 4v_{out}(t) = 0.3v_{in}(t) - 4x_1(t)
\]

after substituting for \( v_{out} \). This is equivalent to the integral equation
\[-x_2(t) = -\int_{-\infty}^{t} \dot{x}_2(\tau) d\tau = -0.3 \int_{-\infty}^{t} v_{in}(\tau) d\tau - 4 \int_{-\infty}^{t} [-x_1(\tau)] d\tau\]

This equation can be realized using the op amp circuit below.

**Remarks:**

1. Overall the realization requires 5 op amps. This can be reduced to 4. Redo the above development so as to reduce the total number of op amps to 4.

2. My experience with filter design classes is that students who use the observable form above have better luck for a working op amp circuit than those who use the controllable form.