LECTURE 33: Energy Considerations in Magnetically Coupled Circuits

Part 1 Main Points/Summary of Lecture

1. **Passivity Principle:** a passive circuit (or circuit element) cannot deliver average power.

2. **Conclusion 1:** Passivity principle implies \( M = M_{12} = M_{21} \).

3. **Conclusion 2:** For the coupled inductors below, the instantaneous stored energy is:

\[
W(t) = \frac{1}{2} L_1 i_1^2(t) + \frac{1}{2} L_2 i_2^2(t) \pm M_1(t)i_1(t) + M_2(t)i_2(t) \quad \text{(B is + and C is -)}
\]

4. **Conclusion 3:** The coefficient of coupling, \( k \), satisfies

**Qualitatively Speaking:** \( W(t) = \) Individually Stored Energy \( \pm \) Mutual Stored Energy
\[ 0 \leq k = \frac{M}{\sqrt{L_1 L_2}} \leq 1 \]

which measures the \% of maximum possible coupling achieved by the coil construction. Transformers using iron cores generally have a coefficient of coupling approximately 1 meaning that they achieve the maximum possible degree of coupling given their physical construction.

**Part 2. Two Examples**

**Example 1.** \( i(t) = 2 \cos(2t) \) A. Find \( W(t) \) and \( W_{\text{max}} \).

![Position B Analysis](image)

Position B Analysis:

\[
W(t) = \frac{1}{2} 4i^2(t) + \frac{1}{2} 5i^2(t) + \frac{1}{2} 6i^2(t) + M i^2(t) \\
= \left(2 + 2.5 + 3 + 3\right) 4 \cos^2(2t) = 42 \cos^2(2t) \text{ J} \\
W_{\text{max}} = 42 \text{ J}
\]

**Exercise.** Repeat for position C.

**Example 2.** A lab experiment to determine the coupling inductance.
Student Measurements:

(i) Primary L: \( L_1 = 4 \) H

(ii) Secondary L: \( L_2 = 9 \) H

(iii) Series Aiding Configuration:

\[
L_{eq} = L_1 + L_2 + 2M = 19 \text{ H} \quad \Rightarrow \quad M = 3 \text{ H} \quad \Rightarrow \quad k = \frac{M}{\sqrt{L_1L_2}} = \frac{3}{6} = 0.5
\]

Part 3. Use passivity assumption to show \( M_{12} = M_{21} = M \) heuristically speaking.
Step 1. Suppositions: \( i_1(t) = \sin(t) \) A and \( i_2(t) = \cos(t) \) A.

Conclusions:

\[
v_1 = L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} = L_1 \cos(t) - M_{12} \sin(t)
\]

and

\[
v_2 = M_{21} \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = M_{21} \cos(t) - L_2 \sin(t)
\]

Step 2. Compute the instantaneous power delivered to the coils:

\[
p(t) = v_1 i_1 + v_2 i_2
\]

\[
= L_1 \cos(t) \sin(t) - M_{12} \sin^2(t) + M_{21} \cos^2(t) - L_2 \sin(t) \cos(t)
\]

\[
= \left( L_1 - L_2 \right) \sin(t) \cos(t) - M_{12} \sin^2(t) + M_{21} \cos^2(t)
\]

\[
= \left( \frac{L_1 - L_2}{2} \right) \sin(2t) - \frac{M_{12}}{2} \left( 1 - \cos(2t) \right) + \frac{M_{21}}{2} \left( 1 + \cos(2t) \right)
\]
Step 3. Compute average power noting that the average value of the integral of \( \sin(2t) \) and \( \cos(2t) \) over one period is zero. Thus, by the passivity principle

\[
P_{\text{ave}} = \frac{1}{T} \int_{0}^{T} p(\tau) d\tau = 0.5 \left[ M_{21} - M_{12} \right] \geq 0
\]

Conclusion: \( M_{21} \geq M_{12} \).

Step 4. A symmetric argument implies that \( 0.5 \left[ M_{12} - M_{21} \right] \geq 0 \) in which case \( M_{12} \geq M_{21} \).

Step 5. Combining the conclusions of steps 3 and 4 yield: \( M_{12} = M_{21} = M \) as we have assumed all along.

Part 4. Stored Energy in coupled inductors:

\[
W(t) = \frac{1}{2} L_{1} i_{1}^{2}(t) + \frac{1}{2} L_{2} i_{2}^{2}(t) \pm M i_{1}(t)i_{2}(t) \quad (+ \text{in B and } - \text{in C})
\]
Derivation with dot at B:

\[ W(t) = \int_{0}^{t} p(\tau) \, d\tau = \int_{0}^{t} \left[ v_1(\tau)i_1(\tau) + v_2(\tau)i_2(\tau) \right] d\tau \]

\[ = \int_{0}^{t} \left[ L_1i_1 \frac{di_1}{d\tau} + Mi_1 \frac{di_2}{d\tau} \right] d\tau + \int_{0}^{t} \left[ Mi_1 \frac{di_1}{d\tau} + Mi_2 \frac{di_2}{d\tau} \right] d\tau \]

\[ = \int_{0}^{t} L_1i_1 \, di_1 + \int_{0}^{t} L_2i_2 \, di_2 + \int_{0}^{t} Mi_1(i_1i_2) \]

(noting that \( di_1i_2 = i_2 \frac{di_1}{d\tau} + i_1 \frac{di_2}{d\tau} \))

\[ = \frac{1}{2} L_1i_1^2(t) + \frac{1}{2} L_2i_2^2(t) + Mi_1(t)i_2(t) \]

**Part 5.** Derivation of the coupling coefficient, \( 0 \leq k = \frac{M}{\sqrt{L_1L_2}} \leq 1 \).

**Exercise.** Look this up and do it.