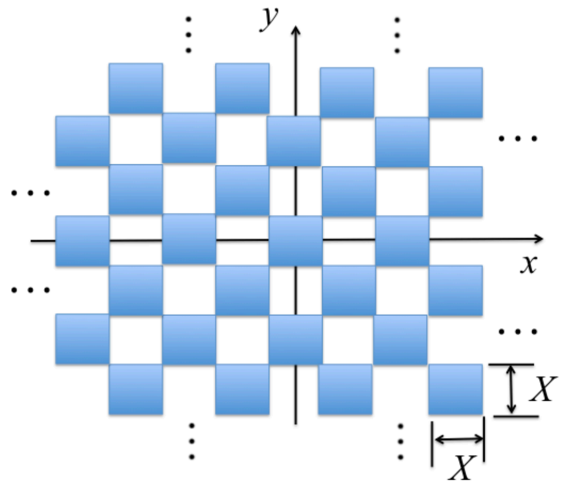
1 December 2021 **Name:____________________**

ECE 638 **Exam No. 2** **Fall 2021**

This is a take-home exam. You may use whatever resources you have at your disposal. However you may **not** consult **anyone** else about any aspect of the exam. You have 48 hours to work the following four problems that are worth a total of 170 pts. (Please see the last page of the exam for a summary of the point assignment.) To obtain maximum partial credit, be sure to show the complete derivation of your answers. Also, please be sure to include any code that you use to solve any problem. Please e-mail your solution to Min Zhao (zhao724@purdue.edu) by 11:59p EST on Friday 3 December. If you have questions about any aspect of the exam, please send an e-mail to me Professor Allebach (jpallebach@gmail.com). I will endeavor to reply to all e-mails as promptly as possible. If necessary, I will schedule a one-on-one Zoom meeting with you address your questions.

1. (50 pts.) Consider the infinitely periodic pattern for the image $g(x,y)$ shown below, which has value 1 in the blue-shaded areas, and value 0, elsewhere.

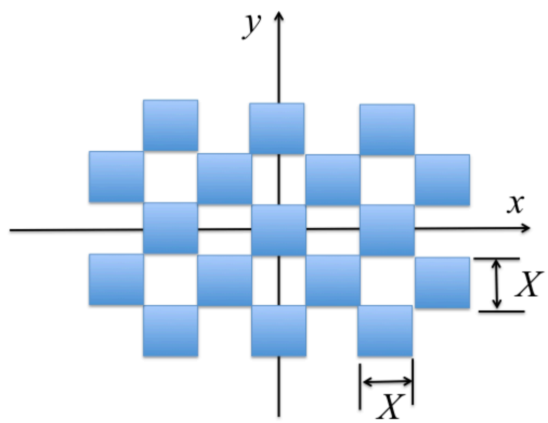


a. (10) Using standard operators and functions, find a simple expression for $g(x,y)$.
Hint: express $g(x,y)$ as the sum of two separate periodic patterns.

b. (10) Based on your answer to part (a) above, find a compact expression for the CSFT $G(u,v)$ of $g(x,y)$.

c. (10) Sketch $G(u,v)$ accurately enough to show that you know what it looks like. Be sure to take into account any cancelation of terms that can be observed.

Now, consider the image $h(x,y)$ with finite extent shown below, which also has value 1 in the blue-shaded areas and value 0, elsewhere.



d. (5) Based on your answer to part (a) above, and using standard operators and functions, find a simple expression for $h(x,y)$.

e. (5) Based on your answers to parts (b) and (d) above, find a compact expression for the CSFT $H(u,v)$ of $h(x,y)$.

f. (10) Based on your answers to parts (c) and (e) above, sketch $H(u,v)$ accurately enough to show that you know what it looks like.

2. (45 pts.) The next two problems deal with a concept known as pulse-width modulation (PWM) with natural sampling. In this problem, we introduce the concept, and establish some notation.

Consider a continuous-parameter, continuous-amplitude signal $f(x)$ that is scaled to lie between 0 and 1, i.e. $0 \le f(x) \le 1$. For simplicity, we will work with functions of one independent variable x, although the extension to functions of x and y is straightforward.

Let us define a threshold signal $t(x)$ according to $t(x) = \text{rep}_X\left[t_0(x)\right]$ where

$$t_0(x) = \begin{cases} \left|\dfrac{2x}{X}\right|, & -X/2 \le x \le X/2 \\ 0, & \text{else} \end{cases}$$

a. (5) Sketch $t(x)$.

Now, let us define the binary-valued PWM signal $g(x)$ according to

$$g(x) = \begin{cases} 1, & f(x) \ge t(x) \\ 0, & \text{else} \end{cases}$$

b. (10) Sketch $g(x)$ for the specific continuous-amplitude signal

$f(x) = \dfrac{1}{2} + \dfrac{1}{2}\cos(2\pi x/(10X)$ for $0 \le x \le 10X$.

Next, let us define the dot profile function

$$p[x;b] = \text{rep}_X\left[p^0[x;b]\right] \tag{2.1}$$

that is the resulting PWM signal $g(x)$ when $f(x)$ has constant value $0 \le b \le 1$, i.e. when $f(x) \equiv b$, then $g(x) = p[x;b]$. Here, $p^0[x;b] = 0, |x| > X/2$.

c. (15) Sketch $p[x;b]$ separately for each case: $b = 0.0, 0.25, 0.5, 0.75,$ and 1.0 for $-X \le x \le X$.

d. (10) Derive an expression for $p^0[x;b]$ in terms of a standard function.

e. (5) Compute the average of the dot profile $p[x;b]$ as a function of b over the interval $-\dfrac{X}{2} \le x \le \dfrac{X}{2}$, i.e.

$$p_{\text{avg}}(b) = \frac{1}{X}\int_{-X/2}^{X/2} p[x;b]\,dx \tag{2.2}$$

Note that PWM with natural sampling is analogous to a continuous-parameter, periodic, clustered-dot halftoning process.

3. (60 pts.) Now, we will use our continuous-parameter PWM process defined in the previous problem to determine a closed form expression for the CSFT $G(u)$ of $g(x)$ in terms of the CSFT $F(u)$ of the input signal $f(x)$ for an arbitrary input $f(x)$. Here again, for simplicity, we will work with functions of just one independent variable x, although the extension to functions of two independent variables x and y is straightforward. The derivation is actually quite tricky, although the final result has a relatively simple interpretation. So I will lead you through it.

   We start with the expression for the dot profile function that we obtained in Problem 2: $p[x;b] = \text{rep}_X\left[p_0[x;b]\right]$. We note that we can also write $g(x) = p[x; f(x)]$ for an arbitrary continuous-amplitude signal $f(x)$.

   Now, let us use the sifting property of the impulse function to write

$$g(x) = \int_s p[x; f(s)]\delta(x-s)ds \qquad (3.1)$$

   Here, we put the dummy variable of integration below the integral sign, since we will soon be working simultaneously with multiple integrals. Next, we consider the Fourier transform $G(u)$ of $g(x)$, which is defined as

$$G(u) = \int_x g(x)e^{-j2\pi ux}dx \qquad (3.2)$$

   Then, we substitute (3.1) into (3.2) and interchange the order of integration to yield

$$G(u) = \int_s \left\{ \int_x p[x; f(s)]\delta(x-s)e^{-j2\pi ux}\, dx \right\} ds \qquad (3.3)$$

   Now comes the trickiest part. Instead directly applying the sifting property of the impulse function to evaluate the integral with respect to x, we consider this integral to be the Fourier transform of the product of two functions that depend on x: $p[x; f(s)]$ and $\delta(x-s)$. Here, we are treating s as having a fixed value. According to the Product Theorem, this will be the convolution of the Fourier transforms of these two functions. Let the variable of integration for this convolution be μ.

   a. (15) Using (2.1) from Problem 2, show that (3.3) can then be written as

$$G(u) = \frac{1}{X}\int_s \left\{ \sum_{k=-\infty}^{\infty} P^0\left[\frac{k}{X}; f(s)\right] \int_\mu \delta(\mu - \frac{k}{X})e^{-j2\pi(u-\mu)s}\, d\mu \right\} ds \qquad (3.4)$$

   Here, $P^0[u, f(s)]$ is the Fourier transform of $p^0[x; f(s)]$ with respect to x. Note that for fixed s, $p[x; f(s)]$ is periodic in x with period X.

b. (10) Now we are ready to apply the sifting property of the impulse function. Applying this property to the evaluation the integral with respect to μ, and then integrating with respect to s, we obtain

$$G(u) = \frac{1}{X} \sum_{k=-\infty}^{\infty} F_k\left(u - \frac{k}{X}\right) \tag{3.5}$$

where $F_k(u)$ is the CSFT of $f_k(x) = P^0\left[\frac{k}{X}; f(x)\right]$ with respect to x. Carry out this derivation to obtain (3.5), and show the steps you used.

The rest of this problem concerns the interpretation of (3.5). First, we note that for fixed k, $f_k(x)$ is the result of processing $f(x)$ through the nonlinear transform $P^0\left[\frac{k}{X}; b\right]$ where the input variable is b.

c. (15) For the dot profile function that you determined in Problem 2, determine expressions for, and sketch, $P^0\left[\frac{k}{X}; b\right]$ for $k = 0,1,2,\text{and } 3$. Discuss the relation between (2.2) from Problem 2 and the characteristics of $P^0\left[0; b\right]$.

d. (15) Suppose that $f(x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x / (10X)$. Sketch $f_k(x)$ for $k = 0,1,\text{and } 2$.

e. (5) Suppose we wish to recover $f(x)$ from $g(x)$ by simply filtering $g(x)$ with an ideal low-pass filter that rejects all frequencies for which $|u| > \frac{1}{2X}$. Discuss how well this will work, in terms of the contributions of the $k = 0$ term and the aliases for which $|k| > 0$ in the expansion given by (3.5).

Please note that the sketches for all parts of this problem only need to be sufficiently accurate to convince one that you know what the function looks like.

4. (30 pts.) Consider the discrete-parameter, continuous-tone image $f(m,n)$ shown below

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Compute the halftone image $g(m,n)$ corresponding to $f(m,n)$ by error diffusion when $f(m,n)$ is processed by starting in the upper left corner and proceeding to the lower right corner row-by-row, and then column-by column. Use the following error diffusion kernel:

X	$\frac{1}{2}$
$\frac{1}{2}$	0

where the X denotes the pixel currently being processed. Your solution should provide a separate 4×4 modified continuous-tone image and 4×4 halftone image array for each of the 16 stages in the process of binarizing the 4×4 pixel image $f(m,n)$. You may do this manually, or by writing a simple program in Matlab, C, or Python. If you choose to write a program, please turn in your code with your solution to the problem. When you reach the right edge of the image, you may discard any error terms that are diffused beyond the boundary of the image.

1.	_______	(out of 50 pts.)
2.	_______	(out of 45 pts.)
3.	_______	(out of 60 pts.)
4.	_______	(out of 30 pts.)
Total	_______	(out of 185 pts.)