19 April 2023

ECE 438 Essential Definitions and Relations

CT Metrics

1. Energy $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

2. Power $P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

3. root mean squared value $x_{rms} = \sqrt{P_x}$

4. Area $A_x = \int_{-\infty}^{\infty} x(t) dt$

5. Average value $x_{avg} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$

6. Magnitude $M_x = \max_{-\infty < t < \infty} |x(t)|$

DT Metrics

1. Energy $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

2. Power $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

3. root mean squared value $x_{rms} = \sqrt{P_x}$

4. Area $A_x = \sum_{n=-\infty}^{\infty} x[n]$

5. Average $x_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$

6. Magnitude $M_x = \max_{-\infty < n < \infty} |x[n]|$

CT Signals and Operators

1. $\mathrm{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \end{cases}$

2. $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$

3. $\delta(t) = \lim_{\Delta\to 0} \frac{1}{\Delta} \mathrm{rect}\left(\frac{t}{\Delta}\right)$

4. $x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau) y(\tau) d\tau$

5. $\mathrm{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$

6. $\mathrm{comb}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$

Properties of the CT Impulse

1. $\int_{-\infty}^{\infty} \delta(t) dt = 1$ and $\delta(t) = 0,\ t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t-t_0)\, x(t) dt = x(t_0)$,

   provided $x(t)$ is continuous at $t = t_0$

3. $x(t)\delta(t-t_0) \equiv x(t_0)\delta(t-t_0)$

4. $\delta(t-t_0) * x(t) = x(t-t_0)$

5. $\delta(at-t_0) \equiv \frac{1}{|a|}\delta(t-t_0/a)$

CTFT

Sufficient conditions for existence

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

Transform Relations

1. Linearity

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$$

2. Scaling and shifting

$$x\left(\frac{t - t_0}{a}\right) \overset{\text{CTFT}}{\leftrightarrow} |a| X(af) e^{-j2\pi f t_0}$$

3. Modulation

$$x(t) e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} X(f - f_0)$$

4. Reciprocity

$$X(t) \overset{\text{CTFT}}{\leftrightarrow} x(-f)$$

5. Parseval

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \, \mathrm{d}f$$

6. Initial Value

$$X(0) = \int_{-\infty}^{\infty} x(t) \, \mathrm{d}t$$

$$\mathrm{x}(0) = \int_{-\infty}^{\infty} X(f) \, \mathrm{d}f$$

7. Convolution

$$x(t) * y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) Y(f)$$

8. Product

$$x(t) y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) * Y(f)$$

9. Transform of periodic signal

$$\mathrm{rep}_{\mathrm{T}}[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\mathrm{T}} \mathrm{comb}_{\frac{1}{\mathrm{T}}}[X(f)]$$

10. Transform of sampled signal

$$\mathrm{comb}_{\mathrm{T}}[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\mathrm{T}} \mathrm{rep}_{\frac{1}{\mathrm{T}}}[X(f)]$$

Transform Pairs

1. $\mathrm{rect}(t) \overset{\text{CTFT}}{\leftrightarrow} \mathrm{sinc}(f)$

2. $\delta(t) \overset{\text{CTFT}}{\leftrightarrow} 1$ and $1 \overset{\text{CTFT}}{\leftrightarrow} \delta(f)$

3. $e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} \delta(f - f_0)$

4. $$\cos(2\pi f_0 t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$$

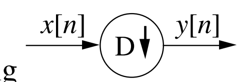
DT Signals and Operators

1. $\mathrm{u}[\mathrm{n}] = \begin{cases} 1, & \mathrm{n} \geq 0 \\ 0, & \mathrm{n} < 0 \end{cases}$

2. $\delta[\mathrm{n}] = \begin{cases} 1, & \mathrm{n} = 0 \\ 0, & \mathrm{n} \neq 0 \end{cases}$

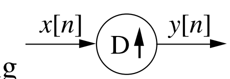
3. $x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[n-k] y[k]$

4. Downsampling



$$y[n] = x[Dn]$$

5. Upsampling



$$y[n] = \begin{cases} x[n/D], & n/D \text{ integer - valued} \\ 0, & \text{else} \end{cases}$$

Properties of the DT Impulse

1. $\sum_{n=-\infty}^{\infty} \delta[n] = 1$ and $\delta[n] = 0, n \neq 0$

2. $\sum_{n=-\infty}^{\infty} \delta[n - n_0] x[n] = x[n_0]$

3. $\delta[n - n_0] * x[n] = x[n - n_0]$

DTFT

Sufficient conditions for existence

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Forward transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \mathrm{d}\omega$$

Transform Relations

1. Linearity

$$a_1 x_1[n] + a_2 x_2[n] \overset{\text{DTFT}}{\leftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega)$$

2. Shifting

$$x[n-n_0] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) e^{-j\omega n_0}$$

3. Modulation

$$x[n] e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} X(\omega - \omega_0)$$

4. Parseval

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega$$

5. Initial Value

$$X(0) = \sum_{n=-\infty}^{\infty} x[n]$$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \mathrm{d}\omega$$

6. Convolution

$$x[n] * y[n] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) Y(\omega)$$

7. Product

$$x[n] y[n] \overset{\text{DTFT}}{\leftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega - \mu) Y(\mu) \mathrm{d}\mu$$

8. Downsampling $x[n] \rightarrow \boxed{D\downarrow} \rightarrow y[n]$

$$Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - 2\pi k}{D}\right)$$

9. Upsampling $x[n] \rightarrow \boxed{D\uparrow} \rightarrow y[n]$

$$Y(\omega) = X(D\omega)$$

Transform Pairs

1. $\delta[n] \overset{\text{DTFT}}{\leftrightarrow} 1$

2. $1 \overset{\text{DTFT}}{\leftrightarrow} 2\pi \operatorname{rep}_{2\pi}[\delta(\omega)]$

3. $e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} 2\pi \operatorname{rep}_{2\pi}[\delta(\omega - \omega_0)]$

4. $\cos(\omega_0 n) \overset{\text{DTFT}}{\leftrightarrow}$
$\pi \operatorname{rep}_{2\pi}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

5.
$$x[n] = \begin{cases} 1, & n = 0,1,\ldots,N-1 \\ 0, & \textit{else} \end{cases} \overset{\textit{DTFT}}{\leftrightarrow}$$

$$X(\omega) = \operatorname{psinc}_N(\omega) e^{-j\omega(N-1)/2}$$

where $\operatorname{psinc}_N(\omega) \triangleq \dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$

Relation between CT and DT

1. Time Domain

$$x_{\mathrm{d}}[n] = x_{\mathrm{a}}(nT)$$

2. Frequency Domain

$$X_{\mathrm{d}}(\omega_{\mathrm{d}}) = f_{\mathrm{s}} \operatorname{rep}_{f_{\mathrm{s}}}\left[X_{\mathrm{a}}(f_{\mathrm{a}})\right]\Big|_{f_{\mathrm{a}} = \frac{\omega_{\mathrm{d}} f_{\mathrm{s}}}{2\pi}},$$

where $f_{\mathrm{s}} = \dfrac{1}{T}$

ZT

Forward transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Transform Relations

1. Linearity

$$a_1 x_1[n] + a_2 x_2[n] \overset{\text{ZT}}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

2. Shifting

$$x[n-n_0] \overset{\text{ZT}}{\leftrightarrow} X(z)z^{-n_0}$$

3. Modulation

$$x[n]z_0^n \overset{\text{ZT}}{\leftrightarrow} X(z/z_0)$$

4. Multiplication by time index

$$nx[n] \overset{\text{ZT}}{\leftrightarrow} -z\frac{\mathrm{d}}{\mathrm{d}z}[X(z)]$$

5. Convolution

$$x[n]*y[n] \overset{\text{ZT}}{\leftrightarrow} X(z)Y(z)$$

6. Relation to DTFT
   If $X_{\text{ZT}}(z)$ converges for $|z|=1$, then the DTFT of $x[n]$ exists and

$$X_{\text{DTFT}}(\omega) = X_{\text{ZT}}(z)\Big|_{z=e^{j\omega}}$$

Transform Pairs

1. $\delta[n] \overset{\text{ZT}}{\leftrightarrow} 1$, all z

2. $a^n u[n] \overset{\text{ZT}}{\leftrightarrow} \dfrac{1}{1-az^{-1}}$, $|z|>|a|$

3. $-a^n u[-n-1] \overset{\text{ZT}}{\leftrightarrow} \dfrac{1}{1-az^{-1}}$, $|z|<|a|$

4. $na^n u[n] \overset{\text{ZT}}{\leftrightarrow} \dfrac{az^{-1}}{(1-az^{-1})^2}$, $|z|>|a|$

5. $-na^n u[-n-1] \overset{\text{ZT}}{\leftrightarrow} \dfrac{az^{-1}}{(1-az^{-1})^2}$, $|z|<|a|$

DFT

Forward transform

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

Inverse transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

Transform Relations

1. Linearity

$$a_1x_1[n] + a_2x_2[n] \overset{\text{DFT}}{\leftrightarrow} a_1X_1[k] + a_2X_2[k]$$

2. Shifting

$$x[n-n_0] \overset{\text{DFT}}{\leftrightarrow} X[k]e^{-j2\pi n_0 k/N}$$

3. Modulation

$$x[n]e^{j2\pi nk_0/N} \overset{\text{DFT}}{\leftrightarrow} X[k-k_0]$$

4. Reciprocity

$$X[n] \overset{\text{DFT}}{\leftrightarrow} N\,x[-k]$$

5. Parseval

$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

6. Initial Value

$$X[0] = \sum_{n=0}^{N-1} x[n]$$

$$x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]$$

7. Periodicity

   $x[n+mN] = x[n]$ for all integers m

   $X[k+lN] = X[k]$ for all integers l

8. Relation to DTFT of a finite length sequence

   Let $x_0[n] \neq 0$ only for $0 \leq n \leq N-1$

$$x_0[n] \overset{\text{DTFT}}{\leftrightarrow} X_0(\omega).$$

   Define $x[n] = \displaystyle\sum_{m=-\infty}^{\infty} x_0[n+mN]$,

$$x[n] \overset{\text{DFT}}{\leftrightarrow} X[k],$$

then $X[k] = X_0\left(e^{j2\pi k/N}\right)$.

9. Periodic convolution

$$x[n] \otimes y[n] \overset{DFT}{\leftrightarrow} X[k]Y[k]$$

10. Product

$$x[n]y[n] \overset{DFT}{\leftrightarrow} \frac{1}{N} X[k] \otimes Y[k]$$

Transform Pairs

1. $\delta[n], 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$
$1, 0 \le k \le N-1$

2. $1, 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$
$N\delta[\mathrm{k}], 0 \le k \le N-1$

3. $e^{j2\pi k_0 n/N}, 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$
$N\delta[\mathrm{k} - \mathrm{k}_0], 0 \le k \le N-1$

4. $\cos(2\pi k_0 n/N), 0 \le n \le N-1 \overset{\text{DFT}}{\leftrightarrow}$
$\frac{N}{2}\{\delta[\mathrm{k} - \mathrm{k}_0] + \delta[\mathrm{k} - (\mathrm{N} - \mathrm{k}_0)]\},$
$0 \le k \le N-1$

5.
$e^{j\omega_0 n}, n = 0,1,\ldots,N-1 \overset{DFT}{\leftrightarrow}$
$\mathrm{psinc}_N\left(2\pi k/N - \omega_0\right) e^{-j(2\pi k/N - \omega_0)(N-1)/2}$

Random Signals

1. Probability density function $f_X(x)$

$$P\{a < X \le b\} = \int_a^b f_X(x)dx$$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad f_X(x) \ge 0$$

2. Probability distribution function $F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$F_X(-\infty) = 0 \quad F_X(\infty) = 1 \quad F_X(x)$ is a non-decreasing function of x

Expectation Operator

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

1. Mean $g(x) = x$

2. 2nd moment $g(x) = x^2$

3. Variance $g(x) = (x - E\{X\})^2$

4. Relation between mean, 2nd moment, and variance

$$\sigma_X^2 = E\{X^2\} - (E\{X\})^2$$

Two Random Variables

1. Bivariate probability density function $f_{XY}(x,y)$

$$P\{a < X \le b, c < Y \le d\} = \int_a^b \int_c^d f_{XY}(x,y)dxdy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 \quad f_{XY}(x,y) \ge 0$$

2. Marginal probability density functions $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

3. Linearity of expectation

$$E\{ag(X,Y) + bh(X,Y)\} = aE\{g(X,Y)\} + bE\{h(X,Y)\}$$

4. Correlation of X and Y

$$E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy$$

5. Covariance of X and Y

$$\sigma_{XY}^2 = E\{(X - E\{X\})(Y - E\{Y\})\}$$
$$= E\{XY\} - E\{X\}E\{Y\}$$

6. Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$
$$0 \le |\rho_{XY}| \le 1$$

7. X and Y are independent if and only if

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

8. X and Y are uncorrelated if and only if

$$E\{XY\} = E\{X\}E\{Y\}$$

Independence implies uncorrelatedness. The converse is not true.

Discrete-time random signals

1. Autocorrelation function

$$r_{XX}[m, m+n] = E\{X[m]X[m+n]\}$$

$X[m]$ is wide-sense stationary if and only if mean is constant, and autocorrelation depends only on difference between arguments, i.e.

$$E\{X[n]\} = \mu_X$$
$$r_{XX}[n] = E\{X[m]X[m+n]\}$$

2. Cross-correlation function

$$r_{XY}[m, m+n] = E\{X[m]Y[m+n]\}$$

$X[m]$ and $Y[m]$ are jointly wide-sense stationary if and only if they are individually wide-sense stationary and their cross-correlation depends only on difference between arguments, i.e.

$$r_{XY}[n] = E\{X[m]Y[m+n]\}$$

3. Wide-sense stationary processes and linear systems.

Let $X[n]$ be wide-sense stationary, and suppose that

$$Y[n] = h[n] * X[n] = \sum_m h[n-m]X[m],$$

then $X[n]$ and $Y[n]$ are jointly wide-sense stationary, and

$$\mu_Y = \mu_X \sum_m h[m]$$
$$r_{XY}[n] = h[n] * r_{XX}[n]$$
$$r_{YY}[n] = h[n] * r_{XY}[-n]$$

Model for Speech

Our model for the speech waveform is given by

$$s[n] = v[n] * e[n],$$

where $s[n]$ is the speech waveform, $v[n]$ is the vocal tract response that typically contains one or more peaks corresponding to resonant frequencies, and $e[n]$ is the excitation provided by the glottis.

$$e[n] = \begin{cases} \sum_k \delta[n - kP], & \text{for a voiced phoneme,} \\ \text{white noise,} & \text{for an unvoiced phoneme.} \end{cases}$$

Here P is the pitch period in samples.

Short-Time Discrete Time Fourier Transform (STDTFT)

Two interpretations:

1. DTFT of windowed waveform centered at (fixed) n where $w[n]$ is the window function

$$\tilde{S}(\omega, n) = \sum_k s[k]w[n-k]e^{-j\omega k}$$

2. Response of narrowband filter centered at (fixed) frequency ω, after shifting to base-band. The impulse response of the narrowband filter is $h_\omega[n] = h_0[n]e^{-j\omega n}$. Here $h_0[n]$ is the impulse response of the base-band filter, which is equivalent to the window function $w[n]$ in the first interpretation. We thus have

$$\tilde{S}(\omega,n) = \left(h_\omega[n] * s[n]\right)e^{+j\omega n}$$

The display of the 2-D image $\left|\tilde{S}(\omega,n)\right|$ as a function of ω and n is called a *spectrogram*. Let N be the length of the window function $w[n]$; and let P be the pitch period. If $N \gg P$, the spectrogram is narrowband. If $N \approx P$, the spectrogram is wideband.

Modulated Filter Bank

Consider an L-channel modulated filter bank with input $x[n]$ and output from the ℓ-th channel given by

$$y_\ell[n] = h_\ell[n] * x[n],$$

where

$$h_\ell[n] = h_0[n]e^{-j2\pi n/L}, \ell = 0, ..., L-1.$$

The output of the filter bank is formed by summing the outputs of all L channels

$$y[n] = \sum_{\ell=0}^{L-1} y_\ell[n]$$

A necessary and sufficient condition for perfect reconstruction, i.e. $y[n] \equiv x[n]$ is that the baseband filter impulse response $h_0[n]$ satisfy

$$h_0[n] = \begin{cases} \frac{1}{L}, & n = 0 \\ 0, & n = kL \end{cases}.$$

There are no constraints on $h_0[n]$ for other values of n.

Miscellaneous

1. Geometric Series

$$\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$$

2. N roots $z_k, k = 0,\ldots,N-1$, of a complex constant u_0, i.e. N solutions to

$$z^N - u_0 = 0$$

$$z_k = |u_0|^{1/N} e^{j(2\pi k + \angle u_0)/N}, k = 0,\ldots,N-1$$

3. Trigonometric Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

$$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$