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ECE 438 Essential Definitions and Relations (Exam 1)

CT Metrics

1. Energy $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

2. Power $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} |x(t)|^2 dt$

3. root mean squared value $x_{rms} = \sqrt{P_x}$

4. Area $A_x = \int_{-\infty}^{\infty} x(t) dt$

5. Average value

$x_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} x(t) dt$

6. Magnitude $M_x = \max_{-\infty < t < \infty} |x(t)|$

DT Metrics

1. Energy $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

2. Power $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

3. root mean squared value $x_{rms} = \sqrt{P_x}$

4. Area $A_x = \sum_{n=-\infty}^{\infty} x[n]$

5. Average $x_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$

6. Magnitude $M_x = \max_{-\infty<n<\infty} |x[n]|$

CT Signals and Operators

1. $\text{rect(t)} = \begin{cases} 1, & |t|<1/2 \\ 0, & |t|>1/2 \end{cases}$

2. $\text{sinc(t)} = \frac{\sin(\pi t)}{\pi t}$

3. $\delta(t) = \lim_{\Delta\to 0} \frac{1}{\Delta} \text{rect}\left(\frac{t}{\Delta}\right)$

4. $x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau) y(\tau) d\tau$

5. $\text{rep}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(t-kT)$

6. $\text{comb}_T[x(t)] = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$

Properties of the CT Impulse

1. $\int_{-\infty}^{\infty} \delta(t) dt = 1$ and $\delta(t) = 0,\ t \neq 0$

2. $\int_{-\infty}^{\infty} \delta(t-t_0)\, x(t) dt = x(t_0)$,

provided $x(t)$ is continuous at $t = t_0$

3. $x(t)\delta(t-t_0) \equiv x(t_0)\delta(t-t_0)$

4. $\delta(t-t_0) * x(t) = x(t-t_0)$

5. $\delta(at-t_0) \equiv \frac{1}{|a|}\delta(t-t_0/a)$

CTFT

Sufficient conditions for existence

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Forward transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \mathrm{d}f$$

Transform Relations

1. Linearity

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\leftrightarrow} a_1 X_1(f) + a_2 X_2(f)$$

2. Scaling and shifting

$$x\left(\frac{t - t_0}{a}\right) \overset{\text{CTFT}}{\leftrightarrow} |a| X(af) e^{-j2\pi f t_0}$$

3. Modulation

$$x(t) e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} X(f - f_0)$$

4. Reciprocity

$$X(t) \overset{\text{CTFT}}{\leftrightarrow} x(-f)$$

5. Parseval

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \, \mathrm{d}f$$

6. Initial Value

$$X(0) = \int_{-\infty}^{\infty} x(t) \mathrm{d}t$$

$$\mathrm{x}(0) = \int_{-\infty}^{\infty} X(f) \mathrm{d}f$$

7. Convolution

$$x(t) * y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) Y(f)$$

8. Product

$$x(t) y(t) \overset{\text{CTFT}}{\leftrightarrow} X(f) * Y(f)$$

9. Transform of periodic signal

$$\mathrm{rep}_{\mathrm{T}}[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\mathrm{T}} \mathrm{comb}_{\frac{1}{\mathrm{T}}}[X(f)]$$

10. Transform of sampled signal

$$\mathrm{comb}_{\mathrm{T}}[x(t)] \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\mathrm{T}} \mathrm{rep}_{\frac{1}{\mathrm{T}}}[X(f)]$$

Transform Pairs

1. $\mathrm{rect}(\mathrm{t}) \overset{\text{CTFT}}{\leftrightarrow} \mathrm{sinc}(\mathrm{f})$

2. $\delta(t) \overset{\text{CTFT}}{\leftrightarrow} 1$ and $1 \overset{\text{CTFT}}{\leftrightarrow} \delta(f)$

3. $e^{j2\pi f_0 t} \overset{\text{CTFT}}{\leftrightarrow} \delta(f - f_0)$

4. $\cos(2\pi f_0 t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$

DT Signals and Operators

1. $\mathrm{u}[\mathrm{n}] = \begin{cases} 1, & \mathrm{n} \geq 0 \\ 0, & \mathrm{n} < 0 \end{cases}$

2. $\delta[\mathrm{n}] = \begin{cases} 1, & \mathrm{n} = 0 \\ 0, & \mathrm{n} \neq 0 \end{cases}$

3. $x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[n-k] y[k]$

Properties of the DT Impulse

1. $\sum_{n=-\infty}^{\infty} \delta[n] = 1$ and $\delta[n] = 0, n \neq 0$

2. $\sum_{n=-\infty}^{\infty} \delta[n - n_0] x[n] = x[n_0]$

3. $\delta[n - n_0] * x[n] = x[n - n_0]$

DTFT

Sufficient conditions for existence

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \text{ or } \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Forward transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Inverse transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Transform Relations

1. Linearity

$$a_1 x_1[n] + a_2 x_2[n] \overset{\text{DTFT}}{\leftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega)$$

2. Shifting

$$x[n - n_0] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) e^{-j\omega n_0}$$

3. Modulation

$$x[n] e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} X(\omega - \omega_0)$$

4. Parseval

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

5. Initial Value

$$X(0) = \sum_{n=-\infty}^{\infty} x[n]$$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) d\omega$$

6. Convolution

$$x[n] * y[n] \overset{\text{DTFT}}{\leftrightarrow} X(\omega) Y(\omega)$$

7. Product

$$x[n] y[n] \overset{\text{DTFT}}{\leftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega - \mu) Y(\mu) d\mu$$

Transform Pairs

1. $\delta[n] \overset{\text{DTFT}}{\leftrightarrow} 1$

2. $1 \overset{\text{DTFT}}{\leftrightarrow} 2\pi \, \text{rep}_{2\pi}[\delta(\omega)]$

3. $e^{j\omega_0 n} \overset{\text{DTFT}}{\leftrightarrow} 2\pi \, \text{rep}_{2\pi}[\delta(\omega - \omega_0)]$

4. $\cos(\omega_0 n) \overset{\text{DTFT}}{\leftrightarrow}$

$$\pi \, \text{rep}_{2\pi}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

5.

$$x[n] = \begin{cases} 1, & n = 0, 1, \ldots, N-1 \\ 0, & else \end{cases} \overset{DTFT}{\leftrightarrow}$$

$$X(\omega) = \text{psinc}_N(\omega) \cdot e^{-j\omega(N-1)/2}$$

$$\text{psinc}_N(\omega) \triangleq \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Here, "psinc" means periodic sinc function.

Relation between CT and DT

1. Time Domain

$$x_d[n] = x_a(nT)$$

2. Frequency Domain

$$X_d(\omega_d) = f_s \text{rep}_{f_s}\left[X_a(f_a)\right]\Big|_{f_a = \frac{\omega_d f_s}{2\pi}},$$

$$\text{where } f_s = \frac{1}{T}$$

Miscellaneous

1. Geometric Series

$$\sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}$$

2. N roots $z_k, k = 0, \ldots, N-1$, of a complex constant u_0, i.e. N solutions to

$$z^N - u_0 = 0$$

$$z_k = |u_0|^{1/N} e^{j(2\pi k + \angle u_0)/N}, k = 0, \ldots, N-1$$

3. Trigonometric Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

$$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$