April 17, 1998 **Name:** SOLUTIONS

EE 438 **Exam No. 3** **Spring 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (20 pts.) Consider a random signal $X[n]$ with probability density function

$$f_X(x)=\begin{cases}\frac{3}{4}(1-x^2), & |x|<1\\ 0, & \text{else}\end{cases}$$

Note: The density function is even. Therefore, without doing any computation, it may be concluded that $E\{x\}=0$.

a. (8) Find the mean and variance of X.

The signal $X[n]$ is quantized uniformly so that each sample can be represented with an 8 bit binary word. Let $Y[n]=Q\{X[n]\}$ denote the quantized signal, where $Q\{\cdot\}$ represents the operation of the quantizer.

b. (12) Compute the signal to quantization noise power ratio in decibels for $Y[n]$.

a) $$E\{x\}=\int_{-\infty}^{\infty} x f_x(x)\,dx=\frac{3}{4}\int_{-1}^{1} x(1-x^2)\,dx$$

$$=\frac{3}{4}\left[\frac{1}{2}x^2-\frac{1}{4}x^4\right]_{-1}^{1}=\frac{3}{4}\left\{\left[\frac{1}{2}-\frac{1}{4}\right]-\left[\frac{1}{2}-\frac{1}{4}\right]\right\}=\underline{\underline{0}}$$

$$\sigma_x^2=E\{x^2\}-\left[E\{x\}\right]^2=\frac{3}{4}\int_{-1}^{1}x^2(1-x^2)\,dx \quad (E\{x\}=0)$$

$$=\frac{3}{4}\left[\frac{1}{3}x^3-\frac{1}{5}x^5\right]_{-1}^{1}=\frac{3}{4}\left\{\left[\frac{1}{3}-\frac{1}{5}\right]-\left[-\frac{1}{3}-\left(-\frac{1}{5}\right)\right]\right\}=\underline{\underline{\frac{1}{5}}}$$

b) $$SNR=10\log_{10}\frac{E\{x^2\}}{E\{n_q^2\}}=10\log_{10}\frac{\sigma_x^2}{\frac{\Delta^2}{12}}$$

$\frac{\Delta^2}{12}$ ← Variance of a uniform distribution

Need to find Δ.

(over)

1. (continued)

The dynamic range of the signal is $[-1, 1]$. Since we are using 8 bits to represent each sample, there are 2^8 possible words.

$$\Rightarrow \Delta = \frac{2}{2^8} = \underline{\underline{2^{-7}}}$$

Thus,

$$SNR = 10 \log_{10} \frac{\left(\frac{1}{5}\right)}{\left[\frac{(2^{-7})^2}{12}\right]} = \underline{\underline{45.9 \text{ dB}}}$$

2. (20 pts.) X and Y are two independent random variables with mean 1 and variance 2. We form two new random variables W and Z according to

$$W = 2X + Y$$

$$Z = X + 2Y$$

a. (5) Find the means of W and Z.
b. (5) Find the variances of W and Z.
c. (5) Find the covariance of W and Z.
d. (5) Find the correlation coefficient for W and Z.

a) $E\{W\} = E\{2X+Y\} = 2E\{X\} + E\{Y\} = 3$

$E\{Z\} = E\{X+2Y\} = E\{X\} + 2E\{Y\} = 3$

b) $E\{W^2\} = E\{(2X+Y)(2X+Y)\} = 4E\{X^2\} + 4E\{XY\} + E\{Y^2\}$

[$E\{X^2\} = \sigma_X^2 + (\bar{X})^2 = 2 + (1)^2 = 3$ Similarly, $E\{Y^2\} = 3$
$E\{XY\} = E\{X\}E\{Y\} = 1 \cdot 1 = 1$ Since X + Y are indep.]

$\Rightarrow E\{W^2\} = 4(3) + 4(1) + 3 = 19$

$\Rightarrow \sigma_W^2 = E\{W^2\} - [E\{W\}]^2 = 19 - (3)^2 = 10$

$E\{Z^2\} = E\{X^2\} + 4E\{XY\} + 4E\{Y^2\} = 3 + 4(1) + 4(3) = 19$

$\Rightarrow \sigma_Z^2 = 19 - (3)^2 = 10$

Comment for parts a) and b): (over)
Since X and Y are identically distributed, it can be concluded from the symmetry in the expressions for W and Z, that $E\{W\} = E\{Z\}$ and $E\{W^2\} = E\{Z^2\}$; so it is only necessary to compute one of each pair.

2. (continued)

c)

$$\sigma_{WZ}^2 = E\{(W-\overline{W})(Z-\overline{Z})\} = E\{WZ - W\overline{Z} - \overline{W}Z + \overline{W}\,\overline{Z}\}$$

$$= E\{WZ\} - \overline{Z}E\{W\} - \overline{W}E\{Z\} + \overline{W}\,\overline{Z} = E\{WZ\} - \overline{W}\,\overline{Z}$$

$$E\{WZ\} = E\{(2X+Y)(X+2Y)\} =$$

$$= 2E\{X^2\} + 4E\{XY\} + E\{XY\} + 2E\{Y^2\}$$

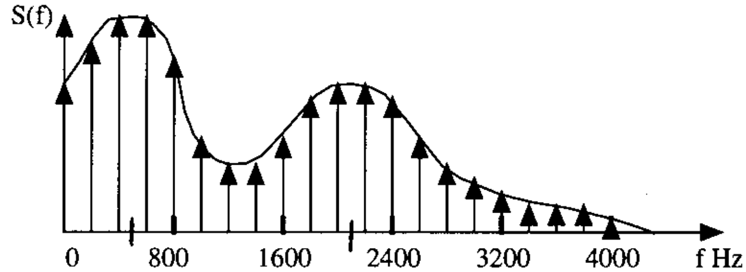
$$= 2(3) + 4(1) + 1 + 2(3) = 17$$

$$\Rightarrow \quad \sigma_{WZ}^2 = 17 - (3)(3) = \underline{8}$$

d)

$$\rho_{WZ} = \frac{\sigma_{WZ}^2}{\sigma_W \sigma_Z} = \frac{8}{\sqrt{10}\sqrt{10}} = \frac{8}{10} = \underline{\frac{4}{5}}$$

3. (30 pts.) A speech signal $s(t)$ consists of a single phoneme. The CTFT of this signal is shown below:



a. (5) Is this voiced or unvoiced speech?

b. (5) What is the pitch period in seconds?

c. (5) What are the formant frequencies?

d. (5) Sketch what a wideband spectrogram of this signal would look like. Be sure to indicate the pitch and formant information.

Suppose we wish to synthesize this signal digitally with a system operating at a sampling frequency of 8 kHz.

e. (5) What is the required interval in samples between excitation pulses.

f. (5) Plot in the complex Z plane the approximate location of the poles for the vocal tract filter.

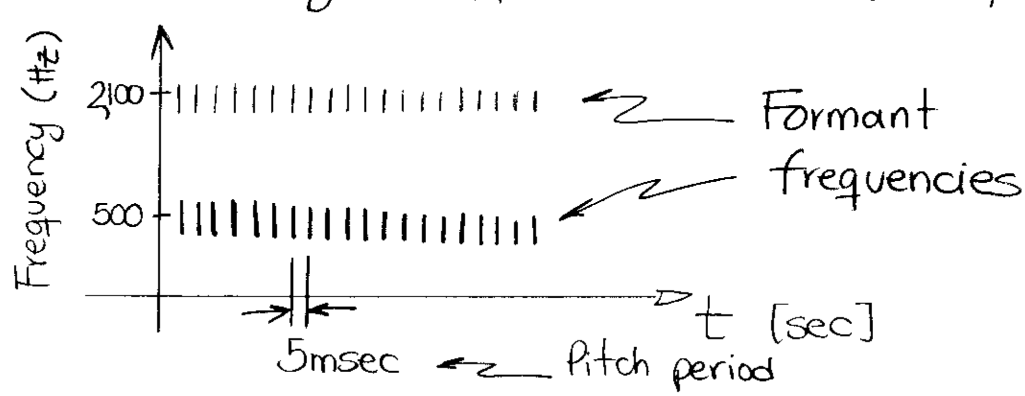
a) Voiced. A train of impulses would produce $S(f)$ as shown.

b) pitch frequency = 200 Hz

$\Rightarrow$ pitch period $= \frac{1}{200} = 5$ msec

c) Approximately $f_1 = 500$ Hz $f_2 = 2,100$ Hz

d)



(over)

3. (continued)

e) $\text{pitch period} = \dfrac{\text{interval in samples}}{f_s}$

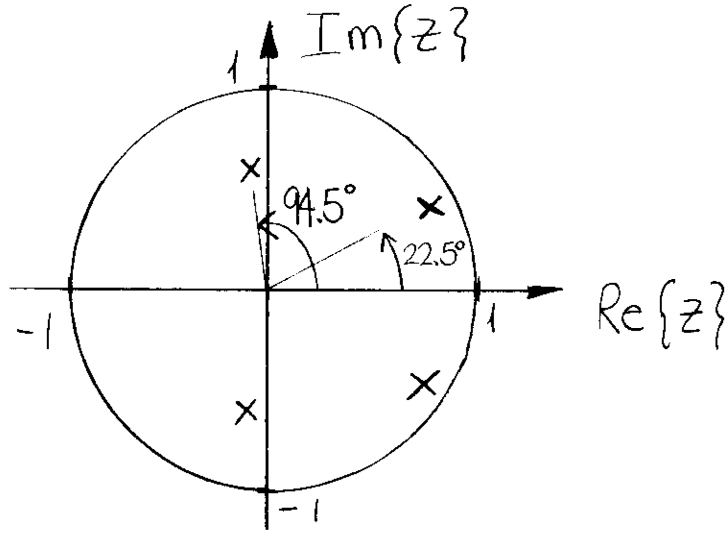
$\Rightarrow$ Interval in samples between excitation pulses $= (8\ \text{kHz})(5\ \text{ms})$

$= 40$

f)

$$\omega_1 = \frac{2\pi f_1}{f_s} = \frac{2\pi(500)}{8000} = \frac{\pi}{8} = 22.5^\circ$$

$$\omega_2 = \frac{2\pi f_2}{f_s} = \frac{2\pi(2{,}100)}{8{,}000} = \frac{21\pi}{40} = 94.5^\circ$$



4. (30 pts.) Your performance in EE 438 has steadily improved during the semester. On the first three exams, you scored 30, 40, and 55, respectively. Based on these grades, you want to predict your score on the final exam. The predictor that you plan to use is of the form $\hat{s}[n] = an + b$, where a and b are constants and $n = 1, 2, 3, 4$ denotes the number of the exam.

   a. (25) Find the coefficients a and b that will yield a minimum mean-squared error prediction, based on the scores that you have attained thus far.

   b. (5) Based on your answer to part a, what score $\hat{s}[4]$ do you predict for the final exam?

a) $E = \frac{1}{3}\sum_{n=1}^{3}\left[\hat{S}[n] - S[n]\right]^2 = \frac{1}{3}\sum_{n=1}^{3}\left[an + b - S[n]\right]^2$

• $\frac{\partial E}{\partial a} = 0 = \frac{1}{3}\sum_{n=1}^{3} 2\left[an + b - S[n]\right]\cdot n$

$= \left\{[a\cdot 1 + b - 30]\cdot 1 + [a\cdot 2 + b - 40]\cdot 2 + [a\cdot 3 + b - 55]\cdot 3\right\}$

$= 14a + 6b - 275 = 0$ —— (Eq 1)

• $\frac{\partial E}{\partial b} = 0 = \frac{1}{3}\sum_{n=1}^{3} 2\left[an + b - S[n]\right]\cdot 1$

$= \left\{[a + b - 30] + [2a + b - 40] + [3a + b - 55]\right\}$

$= 6a + 3b - 125 = 0$ —— (Eq. 2)

Solving (Eq 1) and (Eq 2) $\Rightarrow$

$a = \frac{25}{2} = 12.50$

$b = \frac{50}{3} = 16.67$

Thus, $\hat{S}[n] = \frac{25n}{2} + \frac{50}{3}$

b) $\hat{S}[4] = \frac{25(4)}{2} + \frac{50}{3} = \frac{200}{3} = 66.67$

[redacted]