March 20, 1998 **Name:** SOLUTIONS

EE 438 **Exam No. 2** **Spring 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (30 pts.) Consider the causal system defined by the equation $y[n] = x[n] - \frac{1}{2}y[n-1]$, with input $x[n] = (\frac{3}{4})^n u[-n]$.
   a. (12) Find the Z transform $Y(z)$ of the output $y[n]$.
   b. (6) Sketch a plot of the poles and zeros for $Y(z)$, and show its region of convergence.
   c. (12) Using your expression for $Y(z)$, find the response $y[n]$.

a) $Y(z) = X(z) - \frac{1}{2} Y(z) z^{-1}$

$Y(z)\left[1 + \frac{1}{2}z^{-1}\right] = X(z) \Rightarrow Y(z) = \dfrac{X(z)}{1+\frac{1}{2}z^{-1}}$

ROC: $|z| > \frac{1}{2}$ system is causal

$X(z) = \dfrac{1}{1 - \frac{z}{(\frac{3}{4})}}$ ROC $|z| < \frac{3}{4}$ (compare to HW#4 prob 1d)

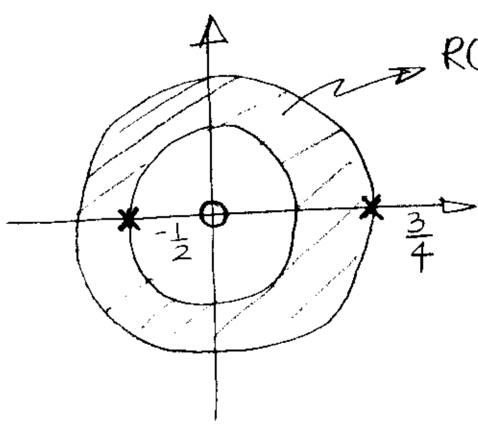
Thus, $Y(z) = \dfrac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{4}{3}z)}$

$= \dfrac{z}{(z+\frac{1}{2})(\frac{3}{4}-z)\cdot\frac{4}{3}}$

b)

Zero: $z = 0$

Poles: $z = -\frac{1}{2}, \frac{3}{4}$



ROC: $\frac{1}{2} < |z| < \frac{3}{4}$

1. (continued)

c) $$Y(z) = \frac{-\frac{3}{4}z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} = \frac{A}{1+\frac{1}{2}z^{-1}} - \frac{B}{1-\frac{3}{4}z^{-1}}$$

$$= \frac{A(1-\frac{3}{4}z^{-1}) - B(1+\frac{1}{2}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} = \frac{(A-B)+(-\frac{3}{4}A-\frac{1}{2}B)z^{-1}}{(\cdot)(\cdot)}$$

$$\left[\begin{cases} A - B = 0 \Rightarrow A = B \\ -\frac{3}{4}A - \frac{1}{2}B = -\frac{3}{4} \Rightarrow -\frac{3}{4}A - \frac{1}{2}A = -\frac{3}{4} \Rightarrow \frac{5}{4}A = \frac{3}{4} \Rightarrow \boxed{A = B = \frac{3}{5}} \end{cases}\right]$$

Thus:

$$Y(z) = \frac{\frac{3}{5}}{1-(-\frac{1}{2})z^{-1}} - \frac{\frac{3}{5}}{1-\frac{3}{4}z^{-1}}$$

and

$$y[n] = \frac{3}{5}\left(-\frac{1}{2}\right)^n u[n] - \frac{3}{5}\left[-\left(\frac{3}{4}\right)^n u[-n-1]\right]$$

Compare to HW#4 prob. 3

$$y[n] = \frac{3}{5}\left(-\frac{1}{2}\right)^n u[n] + \frac{3}{5}\left(\frac{3}{4}\right)^n u[-n-1]$$

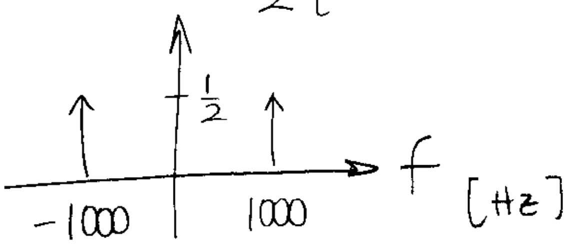
2. (20 pts.) Consider the continuous-time signal $x(t) = \cos(2\pi(1000)t)$.

a. (5) Find and sketch the CTFT $X(f)$.

b. (15) Let $x[n] = x(n/10^4)$. Find and sketch the 20 point DFT $X[k]$ of $x[n], 0 \le n \le 19$.

Hint: Use the formulas wherever they are applicable.

a) $X(f) = \frac{1}{2}\left[\delta(f-1000) + \delta(f+1000)\right]$



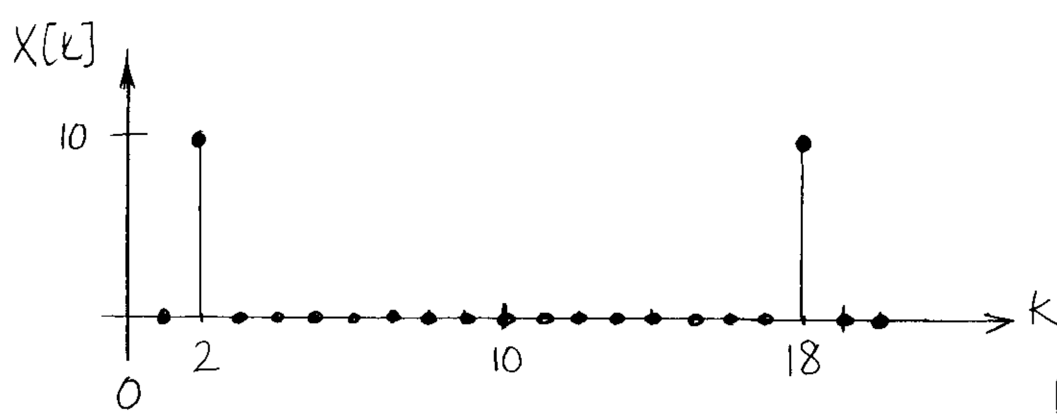
b) $x[n]$ is a <u>sampled</u> version of $x(t)$.
Sampling frequency is: $f_s = 10{,}000$ Hz.

$$x[n] = \cos\left[2\pi(1000)\frac{n}{10{,}000}\right] = \cos\left[\frac{2\pi(2n)}{20}\right]$$

$\updownarrow$ DTF

$$X[k] = \frac{20}{2}\left[\delta(k-2) + \delta(k-(20-2))\right] \qquad 0 \le k < 20$$

$$X[k] = 10\left[\delta(k-2) + \delta(k-18)\right]$$



3. (20 pts.) Consider the length N signal $x[n], 0 \le n \le N-1$. Define another length N signal $y[n] = (-1)^n x[n], 0 \le n \le N-1$.

a. (14) Find a simple expression for the N point DFT $Y[k]$ in terms of the N point DFT $X[k]$.

b. (4) Sketch $Y[k]$ for the case where $x[n] = \cos(\pi n / 5)$, and $N = 20$.

c. (2) Comment on the relation between your answer for parts a and b of Problem 2 and your answer to part b of this problem.

a) Notice that $(-1)^n = e^{j\pi n}$

$$Y[k] = \sum_{n=0}^{N-1} (-1)^n x[n] e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi k n}{N} - \pi n\right)} \longrightarrow e^{-j\frac{2\pi n(k-\frac{N}{2})}{N}}$$

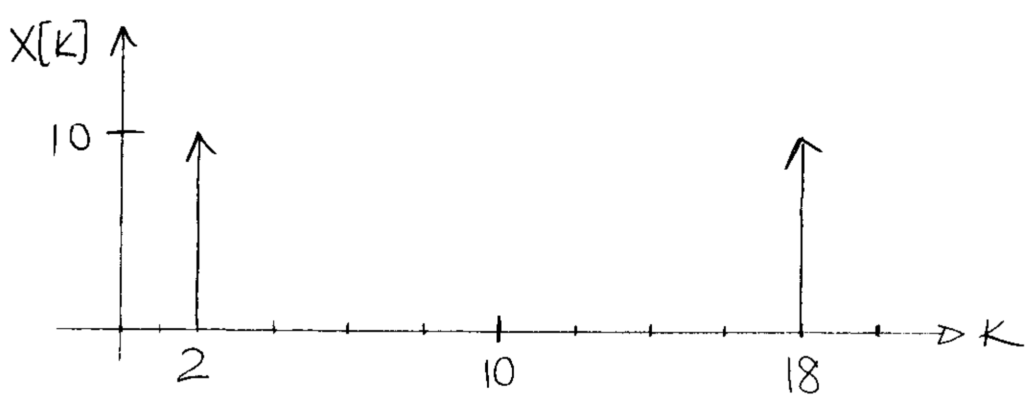
$$= X\left[k - \frac{N}{2}\right]$$

Assuming N is even.

b) $x[n] = \cos\left(\frac{\pi n}{5}\right) = \cos\left(\frac{2\pi 2 n}{20}\right)$ As in prob 2b!

$\updownarrow$ DFT

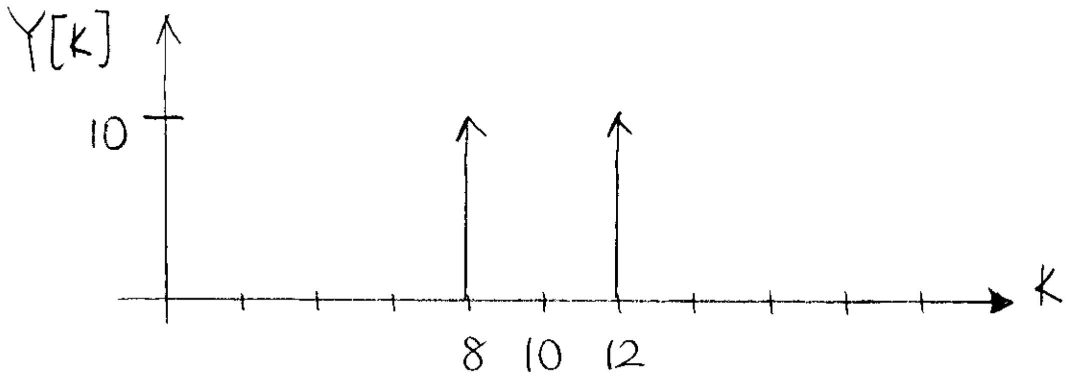
$$X[k] = 10\left[\delta(k-2) + \delta(k-18)\right]$$



3. (continued)

$$Y[k] = X[k-10] = 10\left[\delta\left(((k-10)-2)\right) + \delta\left(((k-10)-18)\right)\right]$$

$$= 10\left[\delta(k-12) + \delta(k-8)\right] \qquad 0 \le k < 20$$



c) This is one way to implement the "fftshift" function of MATLAB, that is, shift DC component to center of spectrum.

N-pt

4. (30 pts.) You have a subroutine that computes the radix 2 FFT, but you wish to compute a DFT $X^{(384)}[k]$ of a signal $x[n]$ that is exactly 384 points in length.

   a. (14) Find an expression for $X^{(384)}[k]$ that shows how it can be computed efficiently by using your radix 2 FFT subroutine.

   b. (10) Based on your answer to part a, draw a flow diagram for the algorithm. (Just show your radix 2 FFT as a single block; don't attempt to break it down.)

   c. (6) Determine the number of complex operations required for this computation.

a) Notice that $384 = 3 \times 128$

So we can compute a 384-pt DFT by combining three 128-pt DFT's. Each 128-pt DFT can be computed efficiently by using a radix-2 FFT. ($128 = 2^7$)

Define:
$$x_0[n] = x[3n+0] \qquad n = 0, \ldots, 127$$
$$x_1[n] = x[3n+1]$$
$$x_2[n] = x[3n+2]$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$X^{(384)}[k] = \sum_{n=0}^{383} x[n] e^{-j\frac{2\pi k n}{384}}$$

$$= \sum_{n=0}^{127} x_0[n] e^{-j\frac{2\pi k(3n+0)}{384}} = X_0^{(128)}[k]$$

$$+ \sum_{n=0}^{127} x_1[n] e^{-j\frac{2\pi k(3n+1)}{384}} = X_1^{(128)}[k]$$

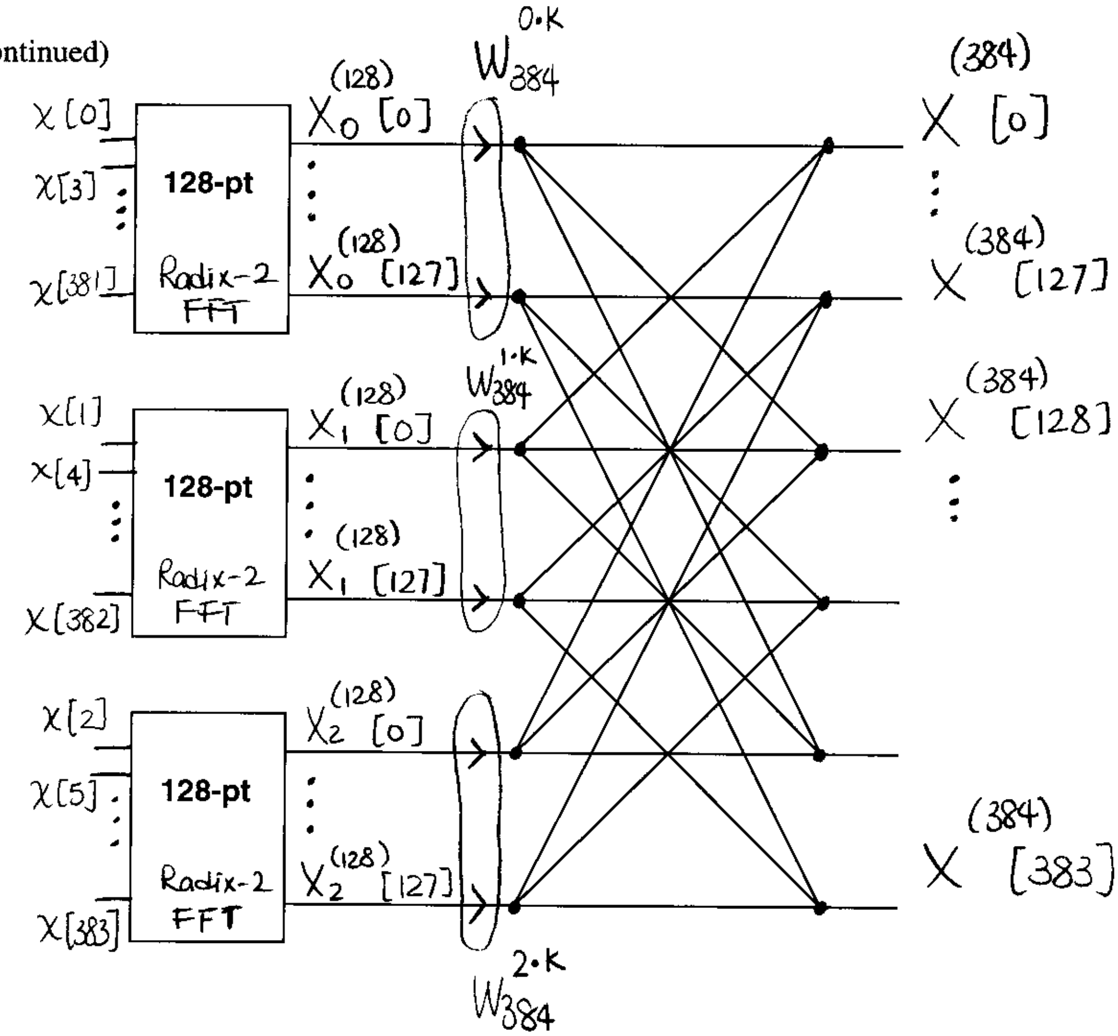
$$+ \sum_{n=0}^{127} x_2[n] e^{-j\frac{2\pi k(3n+2)}{384}} = X_2^{(128)}[k]$$

$$X^{(384)}[k] = W_{384}^{0\cdot k} X_0^{(128)}[k] + W_{384}^{1\cdot k} X_1^{(128)}[k] + W_{384}^{2\cdot k} X_2^{(128)}[k]$$

Use this eq. to draw diagram on next page.

4. (continued)

b)



c) Computation: Recall: 1 complex op = 1 complex add and 1 complex multiply

- Each 128-pt radix-2 FFT : $128 \cdot \log_2 128 = 128 \cdot 7$ complex operations
- For each output point:

$$X^{(384)}[k] = \cancel{W_{384}^{0\cdot k}}^{\,1} X_0^{(128)}[k] + W_{384}^{1\cdot k} X_1^{(128)}[k] + W_{384}^{2\cdot k} X_2^{(128)}[k]$$

two complex operations $\left\{ \begin{array}{l} \text{two adds} \\ \text{two multiplies} \end{array} \right.$

Total: $3 \times (128 \cdot 7) + 384\,(2) = 3{,}456$ co.'s

Compare to: direct implementation: $(384)^2 = 147{,}456$

padding sequence w/ 128 zeros: $512 \cdot \log_2 512 = 4{,}608$