February 13, 1998 **Name:** SOLUTIONS

EE 438 **Exam No. 1** **Spring 1998**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Consider the system defined by the equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

10 a. Find a simple expression for the impulse response $h[n]$ for the system (Do not use Z transforms).

10 b. Find the frequency response $H(\omega)$ for the system, and determine simple expressions for the magnitude and phase of the frequency response.

5 c. Is this system BIBO stable? Justify your answer.

a) Let $x[n] = \delta[n] \Rightarrow y[n] = h[n]$ $\quad h[n] = 0 \quad n<0$

$$h[n] = \delta[n] - \delta[n-1] - h[n-1]$$

$n=0 \quad h[0] = 1 - 0 - 0 = 1$

$n=1 \quad h[1] = 0 - 1 - 1 = -2$

$n=2 \quad h[2] = 0 - 0 - (-2) = 2$

$n=3 \quad h[3] = 0 - 0 - (+2) = -2$

$\vdots$

$$h[n] = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ (-1)^n 2 & n>0 \end{cases}$$

$$\Rightarrow h[n] = 2(-1)^n u[n-1] + \delta[n]$$

b) Let $x[n] = e^{j\omega n} \Rightarrow y[n] = H(\omega)\, x[n]$

$$H(\omega)\cancel{e^{j\omega n}} = \cancel{e^{j\omega n}} - e^{j\omega(\cancel{n}-1)} - H(\omega)\, e^{j\omega(\cancel{n}-1)}$$

$$H(\omega)\left[1 + e^{-j\omega}\right] = 1 - e^{-j\omega}$$

$$\Rightarrow H(\omega) = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j\,\frac{\sin\left(\frac{\omega}{2}\right)}{\cos\left(\frac{\omega}{2}\right)} = j\tan\left(\frac{\omega}{2}\right)$$

$$|H(\omega)| = \left|\tan\left(\frac{\omega}{2}\right)\right| \quad \text{and} \quad \angle H(\omega) = \begin{cases} \frac{\pi}{2}, & \tan\left(\frac{\omega}{2}\right) \geq 0 \\ \frac{\pi}{2} + \pi, & \tan\left(\frac{\omega}{2}\right) < 0 \end{cases}$$

$$j = e^{j\frac{\pi}{2}}$$

c) No, since $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$.

2. (15 pts.) Consider the continuous-time signal

$$x(t) = \begin{cases} \cos[2\pi(8)t], & |t| < 1/2 \\ 0, & \text{else} \end{cases}$$

12 a. Use standard functions and CTFT relations to find the CTFT $X(f)$. (Do *not* directly evaluate any integrals!)

3 b. Sketch $X(f)$.

a) $x(t) = \cos[2\pi(8)t] \cdot \text{rect}(t)$

$\updownarrow$ CTFT

$X(f) = \mathcal{F}\{\cos[2\pi(8)t]\} * \mathcal{F}\{\text{rect}(t)\}$

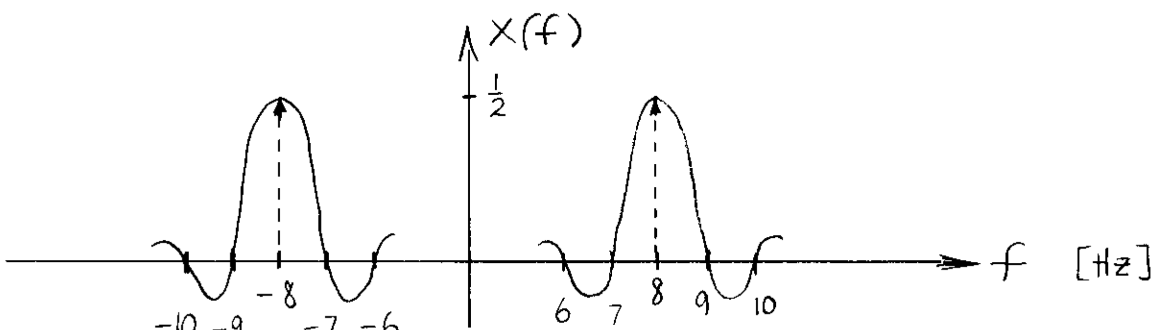
Now, $\cos[2\pi(8)t] \xleftrightarrow{\text{CTFT}} \frac{1}{2}[\delta(f-8) + \delta(f+8)]$

$\text{rect}(t) \xleftrightarrow{\text{CTFT}} \text{sinc}(f)$

Thus, $X(f) = \left\{\frac{1}{2}[\delta(f-8)+\delta(f+8)]\right\} * \{\text{sinc}(f)\}$

$$= \frac{1}{2}\left[\text{sinc}(f-8) + \text{sinc}(f+8)\right]$$

b)



3. (30 pts.) Consider the discrete-time signal

$$x[n] = \begin{cases} 1, & |n| < 16 \\ 0, & \text{else} \end{cases}$$

15 a. Find the DTFT $X(\omega)$ for this signal. Simplify your answer as much as possible.

3 b. Sketch the DTFT $X(\omega)$.

Now let $y[n] = x[n]\cos(\pi n/2)$.

8 c. Use standard transform relations to find the DTFT $Y(\omega)$ for this signal. (Do not evaluate the DTFT sum directly.)

2 d. Sketch the DTFT $Y(\omega)$.

2 e. Comment on the relation between $Y(\omega)$ and the CTFT $X(f)$ that you found in Problem [illegible]. 2

a)

$$X(\omega) = \sum_{n=-15}^{15} e^{-j\omega n} = \sum_{m=0}^{30} e^{-j\omega(m-15)} = e^{j\omega 15} \frac{1 - e^{-j\omega 31}}{1 - e^{-j\omega}} =$$

Change variable $m = n+15$

$$= e^{j\omega 15} \frac{e^{-j\omega\frac{31}{2}}\left[e^{j\omega\frac{31}{2}} - e^{-j\omega\frac{31}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} = \frac{\sin\left(\frac{31\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

c)

$$Y(\omega) = X(\omega) * \left\{\pi \operatorname{rep}_{2\pi}\left[\delta\left(\omega - \tfrac{\pi}{2}\right) + \delta\left(\omega + \tfrac{\pi}{2}\right)\right]\right\}$$

$$= \pi \operatorname{rep}_{2\pi}\left[X\left(\omega - \tfrac{\pi}{2}\right) + X\left(\omega + \tfrac{\pi}{2}\right)\right]$$

e) The discrete-time signal $y[n]$ above is a sampled version of $x(t)$ (from problem 2) if $x[n] = \begin{cases} 1, & |n| \le 16 \\ 0 & \text{else} \end{cases}$

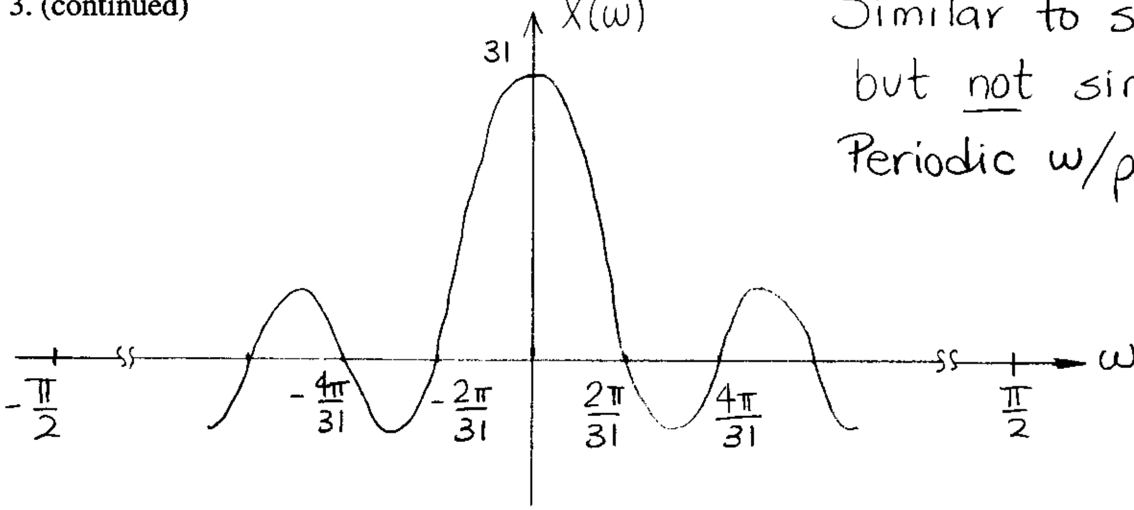
Then, $Y(\omega) = \frac{1}{T_s} \operatorname{rep}_{\frac{1}{T_s}}\left[X(f)\right]\Big|_{f = \frac{\omega f_s}{2\pi}}$ replicates $X(f)$ with period 32, with rescaling.

where: $\frac{\pi}{2} = 2\pi(8)T_s \Rightarrow T_s = \frac{1}{32} \Rightarrow f_s = 32$ (over)

3. (continued)

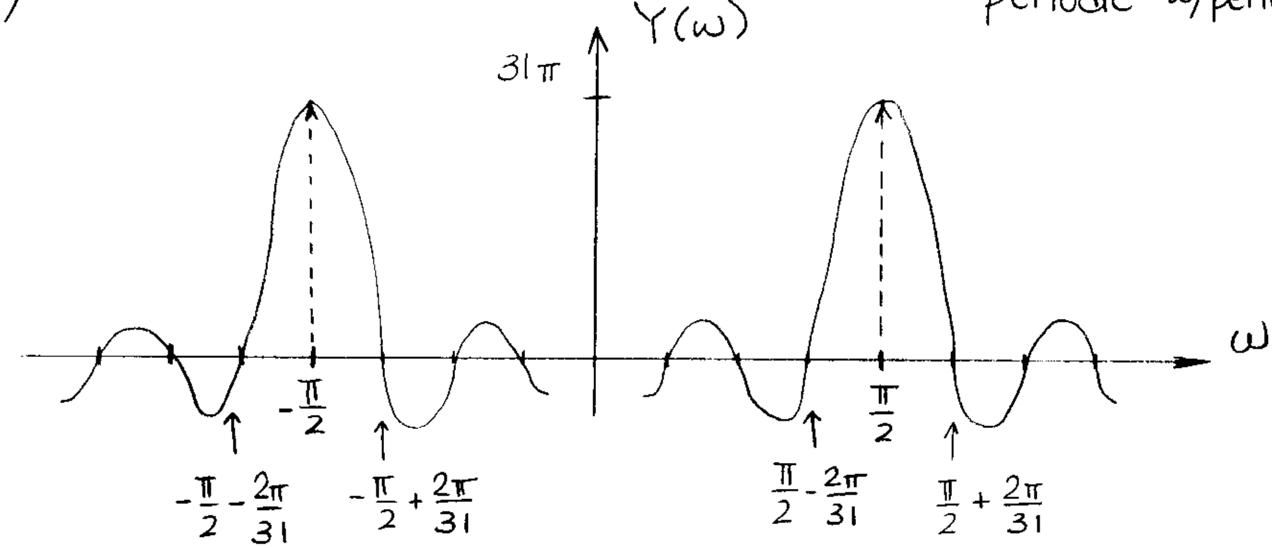
b)

Similar to sinc(·) but <u>not</u> sinc(·). Periodic w/period 2π.

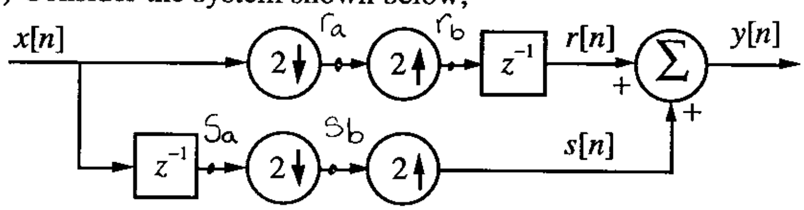


c)

periodic w/period 2π



4. (30 pts.) Consider the system shown below,



where z^{-1} denotes a one-sample delay.

10 a. Find expressions for $r[n]$ and $s[n]$ in terms of $x[n]$.

6 b. Based on your answer to part a, find an expression for $y[n]$ in terms of $x[n]$.

10 c. Find expressions for the DTFTs $R(\omega)$ and $S(\omega)$ in terms of the DTFT $X(\omega)$.

4 d. Based on your answer to part c, find an expression for the DTFT $Y(\omega)$ in terms of the DTFT $X(\omega)$.

a) $r_a[n] = x[2n]$;

$$r_b[n] = \begin{cases} r_a[\frac{n}{2}], & n=2m \\ 0 & \text{else} \end{cases} = \begin{cases} x[n], & n=2m \\ 0 & \text{else} \end{cases}$$

$$r[n] = \begin{cases} x[n-1], & n-1=2m \text{ or } n=2m+1 \\ 0 & \text{else} \end{cases}$$

$$s_a[n] = x[n-1]$$

$$s_b[n] = s_a[2n] = x[2n-1]$$

$$s[n] = \begin{cases} s_b[\frac{n}{2}], & n=2m \\ 0 & \text{else} \end{cases} = \begin{cases} x[2\cdot\frac{n}{2}-1], & n=2m \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} x[n-1], & n=2m \\ 0 & \text{else} \end{cases}$$

b)

s[n]

$$\Rightarrow y[n] = r[n] + s[n] = \begin{cases} x[n-1] & n=2m \\ x[n-1] & n=2m+1 \end{cases} = x[n-1]$$

r[n]

4. (continued)

c) $$R_a(\omega) = \frac{1}{2}\left\{X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega-2\pi}{2}\right)\right\}$$

$$R_b(\omega) = R_a(2\omega) = \frac{1}{2}\left\{X\left(\frac{2\omega}{2}\right) + X\left(\frac{2\omega-2\pi}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{X(\omega) + X(\omega-\pi)\right\}$$

$$R(\omega) = \frac{1}{2}\left\{X(\omega) + X(\omega-\pi)\right\} e^{-j\omega}$$

$$S_a(\omega) = e^{-j\omega} X(\omega)$$

$$S_b(\omega) = \frac{1}{2}\left\{ e^{-j\frac{\omega}{2}} X\left(\frac{\omega}{2}\right) + e^{-j\left(\frac{\omega-2\pi}{2}\right)} X\left(\frac{\omega-2\pi}{2}\right)\right\}$$

$$S(\omega) = S_b(2\omega) = \frac{1}{2}\left\{e^{-j\frac{2\omega}{2}} X\left(\frac{2\omega}{2}\right) + e^{-j\left(\frac{2\omega-2\pi}{2}\right)} X\left(\frac{2\omega-2\pi}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{e^{-j\omega} X(\omega) + e^{-j(\omega-\pi)} X(\omega-\pi)\right\}$$

$$= \frac{1}{2} e^{-j\omega}\left\{X(\omega) - X(\omega-\pi)\right\}$$

d) $$Y(\omega) = R(\omega) + S(\omega) = \frac{1}{2}\left\{X(\omega) + X(\omega-\pi)\right\} e^{-j\omega} + \frac{1}{2}\left\{X(\omega) - X(\omega-\pi)\right\} e^{-j\omega}$$

$$\Rightarrow Y(\omega) = e^{-j\omega} X(\omega)$$ as expected.