May 5, 1997 **Name:** Solution

EE 438 **Final Exam** **Spring 1997**

- You have 120 minutes to work the following 6 problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. However, you may bring with you 4 sheets of formulas handwritten on both sides of one 8.5x11 in. sheet of paper, readable by the unaided eye.
- Calculators are permitted.

1. (20 pts.) Consider the system defined by the difference equation

$$y[n] = x[n] - x[n-1] - y[n-1]$$

   a. Find the impulse response for this system.

   b. Find simple expressions for the magnitude and phase of the frequency response of this system.

   c. Sketch the magnitude and phase of the frequency response.

   d. Is the system BIBO stable?

a) $h(n) = \delta(n) - \delta(n-1) - h(n-1)$

$y(n) = x(n) - x(n-1) - y(n-1)$

$Y(\omega) = X(\omega) - X(\omega)e^{-j\omega} - Y(\omega)e^{-j\omega}$

$Y(\omega)\left[1 + e^{-j\omega}\right] = X(\omega)\left[1 - e^{-j\omega}\right]$

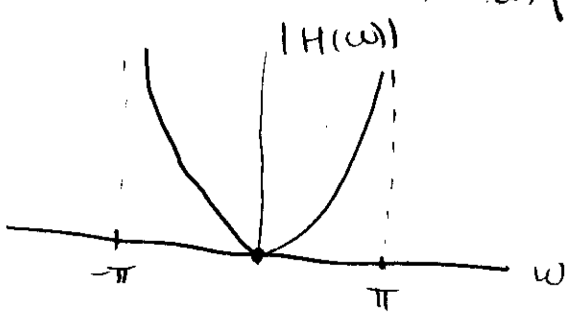
b)

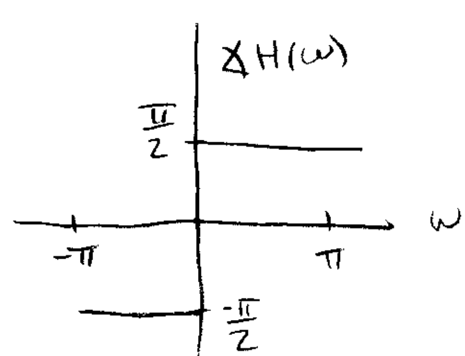
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{\cancel{e^{-j\omega/2}}\left[e^{j\frac{\omega}{2}} - e^{-j\omega/2}\right]}{\cancel{e^{-j\omega/2}}\left[e^{j\omega/2} + e^{-j\omega/2}\right]}$$

$$= \frac{2j\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2) = e^{j\frac{\pi}{2}}\tan\left(\frac{\omega}{2}\right)$$

c) $|H(\omega)| = |\tan(\omega/2)|$

$$\measuredangle H(\omega) = \begin{cases} \frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} - \pi & \omega < 0 \end{cases}$$





d) No, the system is Not Stable.

2. (20 pts.) Consider the signal

$$x[n] = \begin{cases} 2^{-n}, & 0 \le n \le 20 \\ 0, & \text{else} \end{cases}$$

a. Compute the energy for this signal.

b. Compute the power for this signal.

c. Find a simple expression for the DTFT of this signal.

d. Find a simple expression for the 32 point DFT of this signal, using the samples $x[n]$ for $0 \le n \le 31$.

a) energy $= \sum_n [x(n)]^2 = \sum_{n=0}^{20} (2^{-n})^2 = \sum_{n=0}^{20} 2^{-2n} = \sum_{n=0}^{20} \left(\frac{1}{4}\right)^n$

$$= \frac{1 - \left(\frac{1}{4}\right)^{21}}{1 - \frac{1}{4}}$$

b) power $= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \longrightarrow 0$

c) $X(\omega) = \sum_{n=0}^{20} 2^{-n} e^{-j\omega n} = \sum_{n=0}^{20} \left(\frac{1}{2} e^{-j\omega}\right)^n$

$$= \frac{1 - \left(\frac{1}{2} e^{-j\omega}\right)^{21}}{1 - \frac{1}{2} e^{-j\omega}}$$

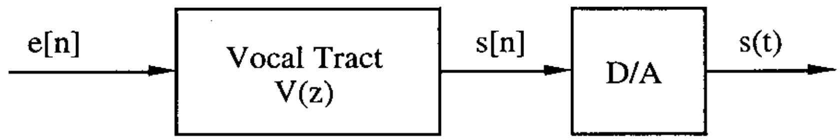
d) $N = 32$

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{32}} = \frac{1 - \frac{1}{2} e^{-j21 \cdot 2\pi k/32}}{1 - \frac{1}{2} e^{-j2\pi k/32}}$$

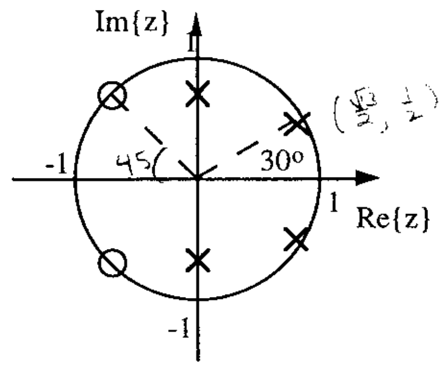
$k = 0, \ldots, 31$

$$= \frac{1 - \frac{1}{2} e^{-j21\pi k/16}}{1 - \frac{1}{2} e^{-j\pi k/16}}$$

3. (20 pts.) The digital synthesizer for voiced speech shown below operates at a 10 kHz sampling rate.



The excitation is given by $e[n] = \sum_{k=-\infty}^{\infty} \delta[n - 200k]$. The vocal tract transfer function $V(z)$ has poles and zeros at the locations shown below:



a. What is the pitch period in seconds?

b. Find the formant frequencies in Hz, and rank them according to their strength, *i.e.* how peaked the vocal tract response is at the corresponding frequency.

c. Sketch what a *wideband* spectrogram would look like for this utterance. Be sure to label the pitch and formant information appropriately.

d. Sketch what a *narrowband* spectrogram would look like for this utterance. Be sure to label the pitch and formant information appropriately.

a) e(n) has a δ-function every 200 n

$P = nT_s = (200) \cdot \frac{1}{10{,}000} = 0.02 \text{ s} = 20 \text{ ms}$

b) formant frequencies ⇒ close to where poles are in V(z)

Poles of V(z): $z = j\sqrt{2}$, $z = -\sqrt{2}j$

$z = \frac{\sqrt{3}}{2} + \frac{1}{2}j$, $z = \frac{\sqrt{3}}{2} - \frac{1}{2}j$

$V(\omega) = V(z)|_{z=e^{j\omega}}$

V(ω) = $e^{j\omega} = e^{j\pi/2}$ ⇒ $\omega = \frac{\pi}{2}$

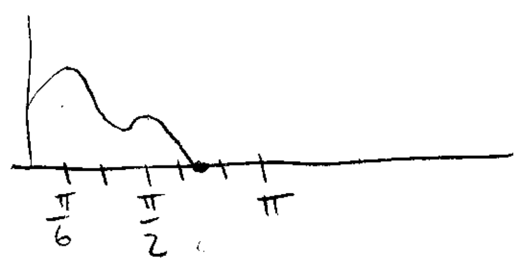
$e^{j\omega} = \sqrt{2}e^{-j\pi/2}$ ⇒ $\omega = -\frac{\pi}{2}$

3. (continued)

$$e^{j\omega} = e^{j\frac{\pi}{6}} \Rightarrow \omega = \pi/6$$
$$e^{j\omega} = e^{j\left(-\frac{\pi}{6}\right)} \Rightarrow \omega = -\pi/6$$

$$\text{Zero at } e^{j\frac{3\pi}{4}} \Rightarrow \omega = 3\pi/4$$

So, $V(\omega)$ looks something like:



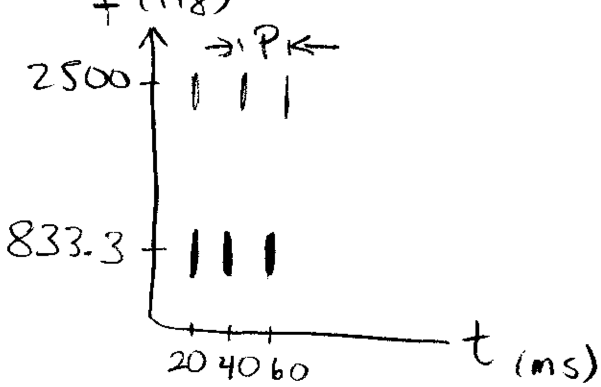
$$\omega = \frac{2\pi f}{f_s}$$

$$\omega_1 = \frac{\pi}{6},\ \omega_2 = \frac{\pi}{2} \Rightarrow f_1 = \frac{\frac{\pi}{6}\cdot 10^4}{2\pi} = \frac{10{,}000}{12} = \boxed{833.3\ \text{Hz}}$$

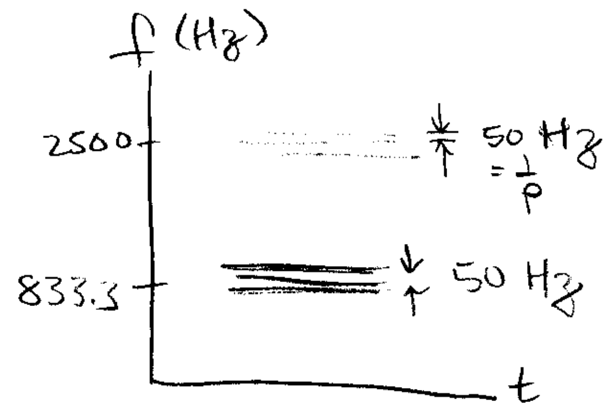
$$f_2 = \frac{\frac{\pi}{2}\cdot 10^4}{2\pi} = \frac{10{,}000}{4} = \boxed{2500\ \text{Hz}}$$

· f_1 is stronger than f_2

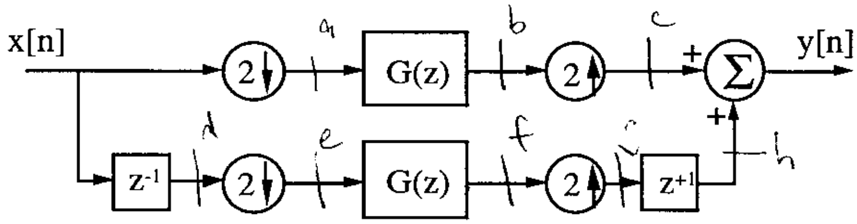
c) wideband spectrogram



d) Narrowband spectrogram



4. (20 pts.) Consider the digital signal processing system shown below.



Find an expression for the overall frequency response $H(e^{j\omega})$ of this system in terms of the frequency response $G(e^{j\omega})$ of the filters in the system. Be sure to simplify your answer as much as possible.

$$X_a(\omega) = \frac{1}{2}\sum_{k=0}^{1} X\left(\frac{\omega-2\pi k}{2}\right) = \frac{1}{2}X\left(\frac{\omega}{2}\right) + \frac{1}{2}X\left(\frac{\omega-2\pi}{2}\right)$$

$$X_b(\omega) = X_a(\omega)\,G(\omega)$$

$$X_c(\omega) = X_b(2\omega) = X_a(2\omega)\,G(2\omega)$$
$$= \left[\frac{1}{2}X(\omega) + \frac{1}{2}X(\omega-2\pi)\right]G(2\omega)$$

$$X_d(\omega) = e^{-j\omega}X(\omega)$$
$$X_e(\omega) = \frac{1}{2}X_d\left(\frac{\omega}{2}\right) + \frac{1}{2}X_d\left(\frac{\omega-2\pi}{2}\right)$$
$$X_f(\omega) = G(\omega)\,X_e(\omega)$$
$$X_G(\omega) = X_f(2\omega) = G(2\omega)\,X_e(2\omega)$$
$$= G(2\omega)\left[\frac{1}{2}X_d(\omega) + \frac{1}{2}X_d(\omega-2\pi)\right]$$
$$= \frac{G(2\omega)}{2}\left[e^{-j\omega}X(\omega) + e^{j(\omega-2\pi)}X(\omega-2\pi)\right]$$

$e^{j2\pi} = 1$

$$X_h(\omega) = e^{+j\omega}X_G(\omega)$$
$$= \frac{G(2\omega)}{2}\left[X(\omega) + \overset{1}{\cancel{e^{-j2\pi}}}X(\omega-2\pi)\right]$$

$$Y(\omega) = X_c(\omega) + X_h(\omega)$$

4. (continued)

$$Y(\omega) = \frac{G(2\omega)}{2}\left[X(\omega) + X(\omega - 2\pi)\right]$$

$$+ \frac{G(2\omega)}{2}\left[X(\omega) + X(\omega - 2\pi)\right]$$

$$= \frac{G(2\omega)}{2}\left[X(\omega) + X(\omega - 2\pi) + X(\omega) + X(\omega - 2\pi)\right]$$

$$= G(2\omega)\left[X(\omega) + X(\omega - 2\pi)\right]$$

$X(\omega) = X(\omega - 2\pi)$ since $X(\omega)$ and $X(\omega - 2\pi$ are periodic with period 2π

$$Y(\omega) = 2G(2\omega)X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 2G(2\omega)$$

5. (20 pts.) The 2-D signal $f(x,y) = \cos[2\pi(90x+10y)]$ is sampled with an ideal sampler at 100 samples/inch to generate the signal

$$f_s(x,y) = \sum_m \sum_n f(0.01m, 0.01n)\delta(x-0.01m, y-0.01n).$$

This signal is then convolved with sinc(100x,100y) to yield the reconstructed signal $f_r(x,y)$.

a. Sketch $f(x,y)$ showing a top view of the x-y plane in which the points where $f(x,y)=1$ are clearly labeled.

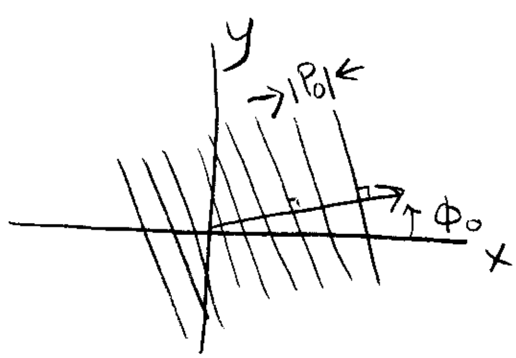
b. Find a simple expression for $f_r(x,y)$.

c. Sketch $f_r(x,y)$ showing a top view of the x-y plane in which the points where $f_r(x,y)=1$ are clearly labeled.

a) $f(x,y) = \cos\left[2\pi(90x+10y)\right]$

$\Phi_0 = \arctan\left(\frac{10}{90}\right) = 6.34°$

$P_0 = \frac{1}{\sqrt{(90)^2+(10)^2}} = 0.011$



b) $f_r(x,y) = \mathrm{sinc}(100x, 100y) ** f_s(x,y)$

$= \mathrm{sinc}(100x, 100y) ** \sum_m \sum_n f(0.01m, 0.01n)\delta(x-\frac{m}{100}, y-\frac{n}{100})$

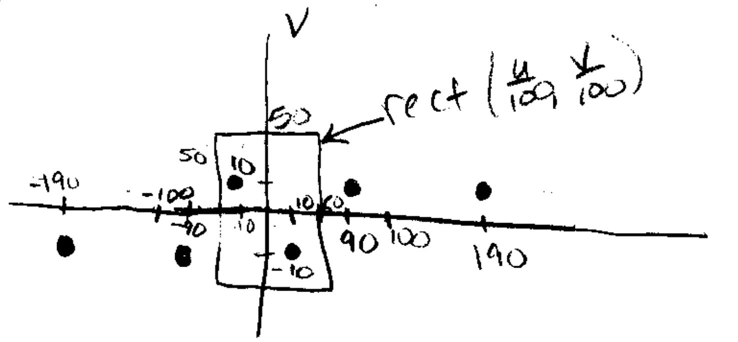
$= \mathrm{sinc}(100x, 100y) ** \left[f(x,y)\,\mathrm{comb}_{\frac{1}{100},\frac{1}{100}}(x,y)\right]$

5. (continued)

$$F_r(u,v) = 10^{-4}\,\mathrm{rect}\left(\frac{u}{100},\frac{v}{100}\right)\cdot F(u,v) ** 10^4\,\mathrm{comb}_{\frac{1}{100},\frac{1}{100}}(u,v)$$

$$= \mathrm{rect}\left(\frac{u}{100},\frac{v}{100}\right)\sum_m \sum_n F(u-m100,\, v-n100)$$

So, $F_r(u,v) =$



$\bullet = \frac{1}{2}$

$$F_r(u,v) = \tfrac{1}{2}\delta(u-10, v+10) + \tfrac{1}{2}\delta(u+10, v-10)$$

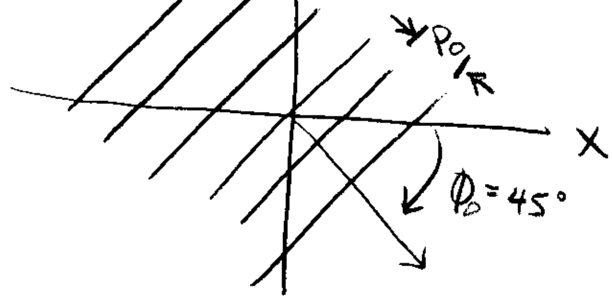
$$f_r(x,y) = \tfrac{1}{2}\cdot e^{j2\pi(10x-10y)} + \tfrac{1}{2}e^{j2\pi(-10x+10y)}$$

$$= \tfrac{1}{2}\left[e^{j2\pi 10(x-y)} + e^{-j2\pi 10(x-y)}\right]$$

$$\boxed{f_r(x,y) = \cos(2\pi 10(x-y))}$$

c)

$f_r(x,y)$



$$P_0 = \frac{1}{\sqrt{10^2 + (-10)^2}}$$

$$= \frac{1}{\sqrt{200}} = 0.071$$

$$\Phi_0 = \arctan\left(\frac{-10}{10}\right)$$

$$= -45°$$

6. (20 pts.) The STDTFT is defined as

$$X(e^{j\omega},n)=\sum_k x[k]w[n-k]e^{-jwk}.$$

Consider the signal

$$x[n]=\begin{cases}\cos(\pi n/6), & n<0\\ \cos(\pi n/2), & n>=0\end{cases},$$

and assume a rectangular window

$$w[n]=\begin{cases}1, & |n|<20\\ 0, & \text{else}\end{cases}. \qquad W(e^{j\omega})=\frac{\sin(20.5\,\omega)}{\sin(\omega/2)}$$

a. Find a simple expression for $X(e^{j\omega},n)$ that is valid for times $n<-20$ and $n>20$.

b. Sketch $|X(e^{j\omega},n)|$ for $n=0$ and for $n=20$. Be sure to label important dimensions.

• if $y(n)=\cos(\omega_0 n)$,

$$X(e^{j\omega},n)=\frac{1}{2}W(e^{j(\omega+\omega_0)})e^{-j(\omega+\omega_0)n}+\frac{1}{2}W(e^{j(\omega-\omega_0)})e^{-j(\omega-\omega_0)n}$$

a) $n<-20 \Rightarrow X(e^{-j\omega},n)=\sum_k x(k)w(n-k)e^{-j\omega k}$

$\Rightarrow$ window entirely in $\cos(\frac{\pi n}{6})$ signal

$$\boxed{X(e^{-j\omega},n)=\frac{1}{2}W(e^{j(\omega+\frac{\pi}{6})})e^{-j(\omega+\frac{\pi}{6})n}+\frac{1}{2}W(e^{j(\omega-\frac{\pi}{6})})e^{-j(\omega-\pi/6)n}}$$

where $W(e^{j\omega})=\sin(20.5\,\omega)/\sin(\omega/2)$

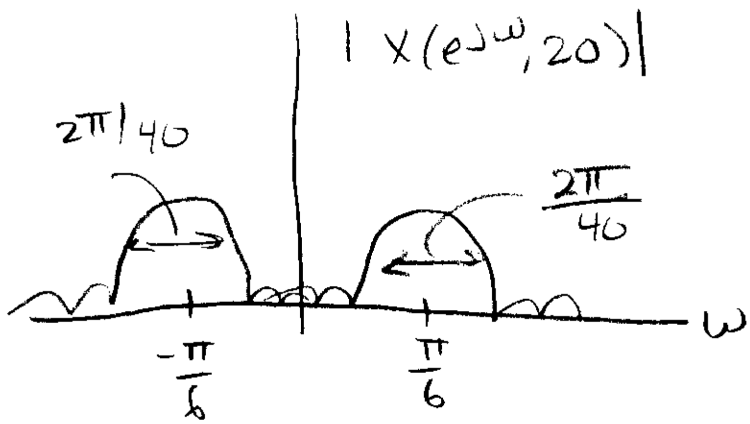
$n>20 \Rightarrow$ window entirely in $\cos(\frac{\pi n}{2})$ signal

$$X(e^{j\omega},n)=\frac{1}{2}W(e^{j(\omega+\frac{\pi}{2})})e^{-j(\omega+\frac{\pi}{2})n}+\frac{1}{2}W(e^{j(\omega-\frac{\pi}{2})})e^{-j(\omega-\frac{\pi}{2})n}$$

6. (continued)

1.	_______
2.	_______
3.	_______
4.	_______
5.	_______
6.	_______
Total	_______

b) $n = 20$



$n = 0$