April 17, 1997 **Name:** ~~Solution~~

EE 438 **Exam No. 3** **Spring 1997**

- You have 75 minutes to work the following 5 problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. However, you may bring with you 3 sheets of formulas handwritten on both sides of one 8.5x11 in. sheet of paper, readable by the unaided eye.
- Calculators are permitted.

1. (20 pts.) Consider the signal $g(x,y) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \text{sinc}[(x-2k)/0.2,(y-2l)/0.2]$.

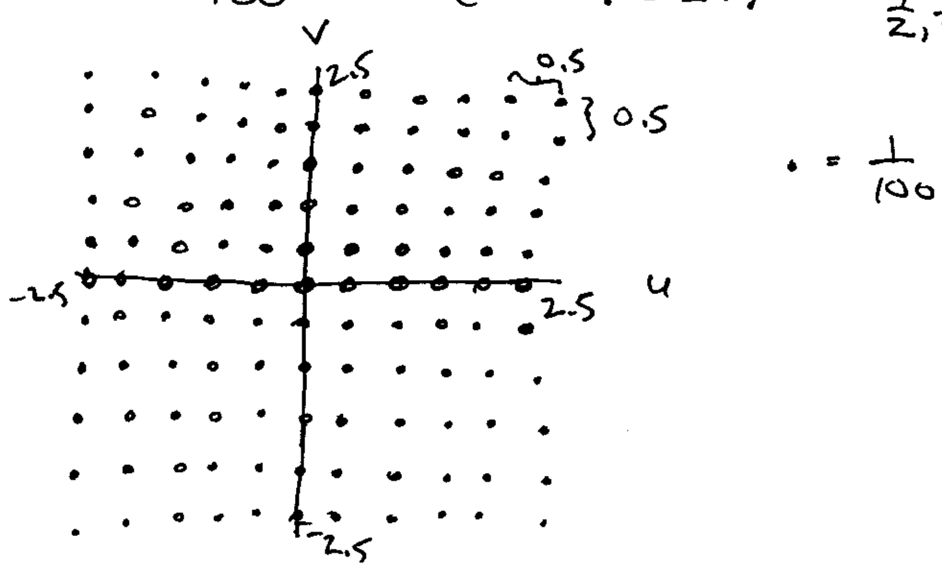
   a. Find a *simple* expression for the 2D continuous-space Fourier transform (CSFT) of $g(x,y)$.

   b. Accurately sketch $G(u,v)$.

$$g(x,y) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \text{sinc}\left[\frac{x-2k}{0.2}, \frac{y-2l}{0.2}\right]$$

$$= \text{sinc}\left(\frac{x}{0.2}, \frac{y}{0.2}\right) ** \text{comb}_{2,2}(x,y)$$

$$G(u,v) = \frac{1}{100}\,\text{rect}(0.2u, 0.2v)\,\text{comb}_{\frac{1}{2},\frac{1}{2}}(u,v)$$



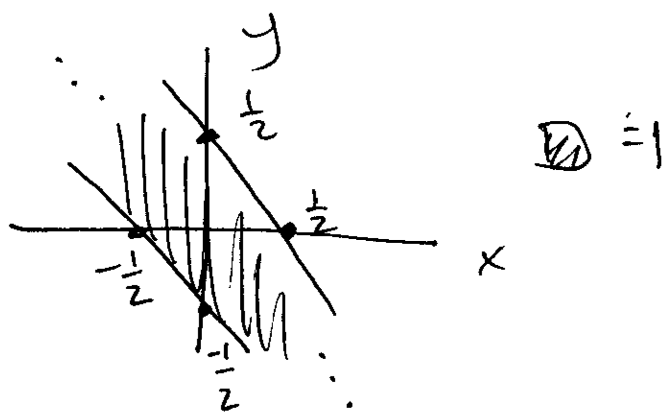
2. (20 pts.) Consider a 1-D function $f(x)$ with 1-D CSFT $F(u)$. Let us define a new 2-D function $g(x,y)$ in terms of $f(x)$ according to $g(x,y) = f(x+y)$.

   a. For the specific function $f(x) = \mathrm{rect}(x)$, sketch $g(x,y)$.

   b. For a general function $f(x)$, find a *simple* expression for the 2-D CSFT $G(u,v)$ of $g(x,y)$ in terms of $F(u)$.

a) $f(x) = \mathrm{rect}(x)$

$$g(x,y) = f(x+y) = \mathrm{rect}(x+y) = \begin{cases} 1 & |x+y| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$$



b)
$$G(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \int\int f(x+y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \int \left[\int f(x+y)\, e^{-j2\pi ux}\, dx \right] e^{-j2\pi vy}\, dy$$

$$= \int F(u) e^{+j2\pi uy}\, e^{-j2\pi vy}\, dy$$

$$= F(u) \cdot \delta(v-u)$$

3. (20 pts.) Consider the digital image $f[m,n]$ shown below.

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{matrix}$$

a. Find the result of filtering this image with the filter $h[k,l]$ given by

		$k = -1$	$k = 0$	$k = 1$
l	-1	$-1/8$	$1/2$	$-1/8$
	0	$-1/4$	1	$-1/4$
	1	$-1/8$	$1/2$	$-1/8$

b. Find a *simple* expression for the frequency response (2D-DSFT) $H(\mu, \nu)$ for this filter. Plot $H(\mu, \nu)$ along the μ and ν axes.

c. Discuss the spatial domain properties of this filter, and relate these properties to its frequency response.

a)

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{3}{8} & \frac{9}{8} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 & 1 \end{matrix}$$

OUTPUT IF WE ASSUME OUTSIDE OF IMAGE IS SAME AS EDGE VALUE

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & 3/8 \\ 0 & -\frac{3}{8} & 9/8 & 3/4 & 3/4 & 9/8 \\ 0 & -1/2 & 3/2 & 1 & 1 & 3/2 \\ 0 & -1/2 & 3/2 & 1 & 1 & 3/2 \\ 0 & -3/8 & 9/8 & 3/4 & 3/4 & 9/8 \end{matrix}$$

OUTPUT IF WE ASSUME OUTSIDE OF IMAGE IS 0

3. (continued)

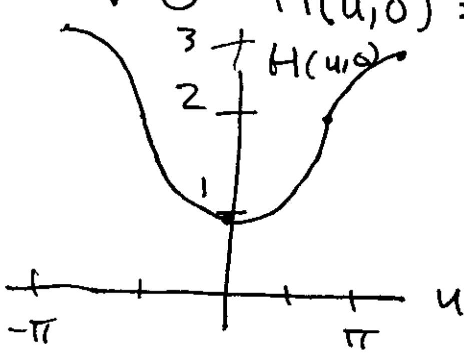
b) $$H(u,v) = \sum_{k=-1}^{1} \sum_{l=-1}^{1} h(k,l)\, e^{-j(ku+lv)}$$

$$= -\frac{1}{8} e^{j(u+v)} - \frac{1}{4} e^{ju} - \frac{1}{8} e^{-j(v-u)} + \frac{1}{2} e^{jv} + 1$$

$$+ \frac{1}{2} e^{-jv} - \frac{1}{8} e^{+j(v-u)} - \frac{1}{4} e^{-ju} - \frac{1}{8} e^{-j(u+v)}$$

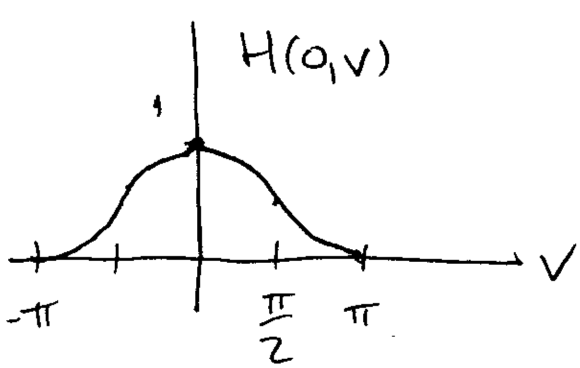
$$= 1 - \frac{1}{2}\cos(u) + \cos(v) - \frac{1}{4}\cos(u+v) - \frac{1}{4}\cos(v-u)$$

$V=0$ $\quad H(u,0) = 2 - \cos(u)$



c) $V=0 \Rightarrow$ Sharpen
$u=0 \Rightarrow$ Smooth

$u=0$ $\quad H(0,v) = \frac{1}{2} + \frac{1}{2}\cos(v)$



4. (20 pts.) If the camera moves while the shutter is open, the image will be blurred. For horizontal motion with constant velocity, such blur can be modeled according to

$$g[m,n] = \sum_{k=0}^{N-1} f[m,n-k], \tag{1}$$

where $f[m,n]$ denotes the desired unblurred image, and $g[m,n]$ denotes the blurred image that is recorded on film (and subsequently digitized). The integer constant N depends on the velocity of the camera and the length of the exposure.

The objective of this problem is to devise a filter for deblurring $g[m,n]$, and thus recovering $f[m,n]$. To do this, we need the 2-D Z transform (ZT) defined as

$$G(z_1,z_2) = \sum_m \sum_n g[m,n] z_1^{-m} z_2^{-n}.$$

a. Use the 2D ZT to derive an expression for the transfer function $H(z_1,z_2)$ corresponding to the blur defined by Eq. (1).

b. From your answer to part a, find a difference equation that will yield as its output $f[m,n]$, when the input is $g[m,n]$.

a) $$G(z_1,z_2) = \sum_m \sum_n \sum_{k=0}^{N-1} f(m,n-k)\, z_1^{-m} z_2^{-n}$$

$$= \sum_{k=0}^{N-1} \sum_m \sum_n f(m,n-k)\, z_1^{-m} z_2^{-n}$$

$$= \sum_{k=0}^{N-1} F(z_1,z_2)\, z_2^{-k}$$

$$H(z_1,z_2) = \frac{G(z_1,z_2)}{F(z_1,z_2)} = \sum_{k=0}^{N-1} z_2^{-k} = \frac{1-\left(\frac{1}{z_2}\right)^N}{1-\frac{1}{z_2}}$$

ASSUMING $\frac{1}{z_2} < 1$

b) $$\frac{1}{H(z_1,z_2)} = \frac{1-\frac{1}{z_2}}{1-\left(\frac{1}{z_2}\right)^N} = \frac{F(z_1,z_2)}{G(z_1,z_2)}$$

- CROSS-MULTIPLY
- INVERSE ZT

$$g(m,n) - g(m,n-1) = f(m,n) - f(m,n-N)$$

5. (20 pts.) Consider 2 random variables X and Y which both have mean 1 and variance 2. Their correlation coefficient is 0.5. We define two new random variables U and V according to

$$U = X + Y$$
$$V = X - Y$$

a. Find their means μ_U and μ_V.

b. Find their variances σ_U^2 and σ_V^2.

c. Find their correlation coefficient ρ_{UV}.

$$E(X^2) = \sigma_X^2 + (E(X))^2 = 2 + 1 = 3$$

$$E(Y^2) = 3$$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} \Rightarrow E(XY) = 2$$

a) $E(U) = E(X+Y) = E(X) + E(Y) = 1 + 1 = \boxed{2}$

$E(V) = E(X-Y) = E(X) - E(Y) = 1 - 1 = \boxed{0}$

b) $$\sigma_U^2 = E(U^2) - (E(U))^2$$
$$= E[(X+Y)(X+Y)] - (2)^2$$
$$= E(X^2) + 2E(XY) + E(Y^2) - 4$$
$$= 3 + 2 \cdot 2 + 3 - 4$$
$$= \boxed{6}$$

$$\sigma_V^2 = E(V^2) - \cancel{(E(V))^2}$$
$$= E[(X-Y)(X-Y)] = E(X^2) - 2E(XY) + E(Y^2)$$
$$= 3 - 2 \cdot 2 + 3 = \boxed{2}$$

c) $$\rho_{UV} = \frac{E(UV) - E(U)E(V)}{\sigma_U \sigma_V} = \frac{E[(X+Y)(X-Y)] - 2 \cdot 0}{\sqrt{6}\sqrt{2}}$$
$$= \frac{1}{\sqrt{12}}\left(E(X^2) - E(Y^2)\right) = \frac{1}{\sqrt{12}}(3-3) = \boxed{0}$$