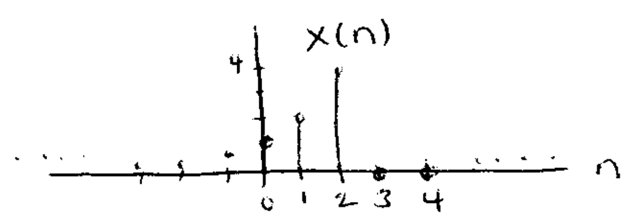
February 13, 1997 **Name:** Solution

EE 438 **Exam No. 1** **Spring 1997**

- You have 75 minutes to work the following five problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. However, you bring with you a sheet of formulas handwritten on both sides of one 8.5x11 in. sheet of paper
- Calculators are permitted.

1. (20 pts.) For the signal $x[n] = 2^n u[-n+2]$, do the following:
   a. Sketch $x[n]$.
   b. State whether or not the signal has finite or infinite duration.
   c. State whether or not the signal is right-sided, left-sided, or two-sided.
   d. State whether or not the signal is causal or anticausal.
   e. Calculate the DTFT $X(e^{j\omega})$ of the signal $x[n]$.

a)



b) infinite duration

c) Left-sided

d) Neither

d)

$$X(\omega) = \sum_{n=-\infty}^{2} 2^n e^{-j\omega n} = \sum_{n=-\infty}^{0} 2^n e^{-j\omega n} + 2^1 e^{-j\omega 1} + 2^2 e^{-j\omega 2}$$

$$= \sum_{m=0}^{\infty} 2^{-m} e^{+j\omega m} + 2e^{-j\omega} + 4e^{-j\omega 2}$$

$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{2} e^{j\omega}\right)^m + 2e^{-j\omega} + 4e^{-j\omega 2}$$

$$= \frac{1}{1 - \frac{1}{2}e^{j\omega}} + 2e^{-j\omega} + 4e^{-j\omega 2} = \frac{4e^{-j\omega 2}}{1 - \frac{1}{2}e^{j\omega}}$$

2. (20 pts.) Consider the system defined by the equation $y[n] = \frac{1}{3}(x[n] - x[n-3])$.

   a. State whether or not the system is linear, ime-invariant, causal, and BIBO stable. (Note: no proofs are needed.)

   b. Find the impulse response $h[n]$ for the system.

   c. Find the frequency response $H(e^{j\omega})$ for the system, and determine simple expressions for the magnitude and phase of the frequency response.

a) Linear
Time Invariant
Causal
BIBO Stable

b) $h(n) = \frac{1}{3}(\delta(n) - \delta(n-3))$

c) $H(\omega) = \frac{1}{3}(1 - e^{-j\omega 3})$

$$= \frac{1}{3} e^{-j\omega 3/2}\left(e^{j\omega 3/2} - e^{-j\omega 3/2}\right) \cdot \frac{2j}{2j}$$

$$= \frac{2j}{3} e^{-j\omega 3/2} \sin(3\omega/2)$$

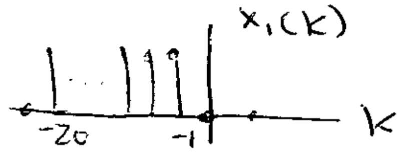
$$= \frac{2}{3} e^{j\left(\frac{\pi}{2} - \frac{\omega 3}{2}\right)} \sin(3\omega/2)$$

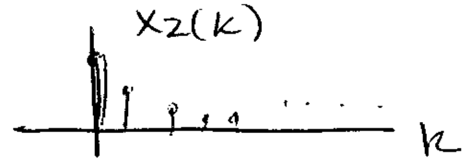
$$|H(\omega)| = \frac{2}{3}|\sin(3\omega/2)|$$

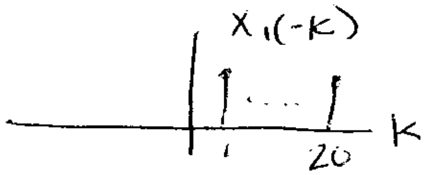
$$\measuredangle H(\omega) = \begin{cases} \frac{\pi}{2} - \frac{\omega 3}{2} + \pi & \sin(3\omega/2) < 0 \\ \frac{\pi}{2} - \frac{3\omega}{2} & \sin(3\omega/2) \geq 0 \end{cases}$$

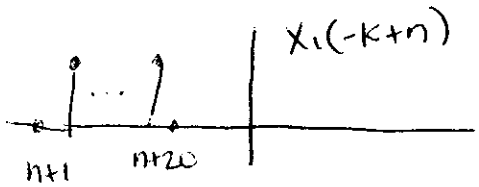
3. (20 pts.) Find the convolution of the following two signals:

$$x_1[n] = u[n+20] - u[n], \;\; x_2[n] = 2^{-n}u[n].$$







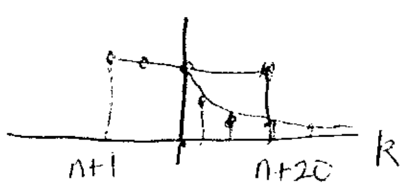


Case 1 No overlap

$n+20 < 0$

$n < -20$ $\qquad y(n) = 0$

Case 2 $x_1(-k+n)$ entering $x_2(k)$



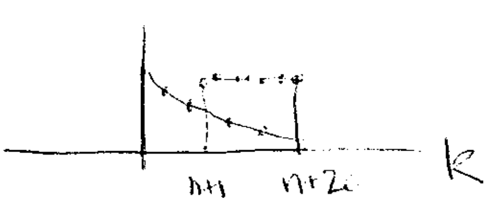
$n+20 \geq 0$ and $n+1 < 0$

$n \geq -20$ $\qquad n < -1$

$-20 \leq n < -1$

$$y(n) = \sum_{k=0}^{n+20} 2^{-n} = \frac{1-\left(\frac{1}{2}\right)^{n+21}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n+21}\right)$$

Case 3 $x_1(-k+n)$ and $x_2(k)$ fully overlap



$n+1 \geq 0$ and $n+20 \geq 0$

$n \geq -1$

3. (continued)

$$y(n) = \sum_{n+1}^{n+20} 2^{-k} \qquad \text{let} \quad m = k-(n+1)$$

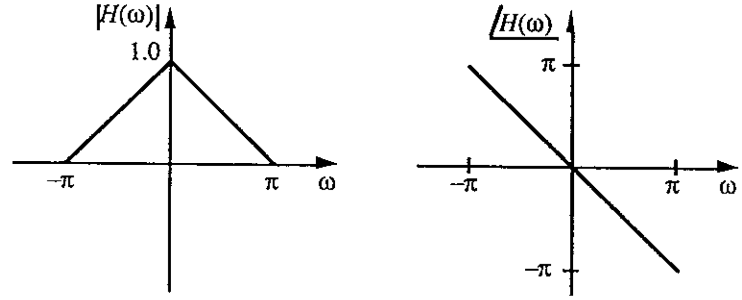
$$= \sum_{m=0}^{19} 2^{-(m+n+1)} = 2^{-n} 2^{-1} \sum_{m=0}^{19} 2^{-m}$$

$$= 2^{-n} 2^{-1} \left(\frac{1-2^{-20}}{1-\frac{1}{2}} \right) = \frac{2^{-n} \cancel{2^{-1}} \left(1-\left(\frac{1}{2}\right)^{20}\right)}{\cancel{2^{-1}}}$$

$$= 2^{-n} \left(1-\left(\tfrac{1}{2}\right)^{20}\right)$$

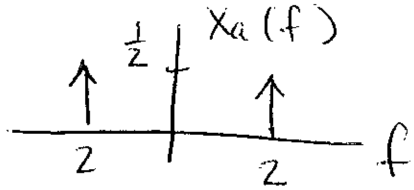
$$y(n) = \begin{cases} 0 & n < -20 \\ 2\left(1-\left(\frac{1}{2}\right)^{n+21}\right) & -20 \le n < -1 \\ 2^{-n}\left(1-\left(\frac{1}{2}\right)^{20}\right) & n \ge -1 \end{cases}$$

4. (20 pts.) The CT signal $x_a(t) = \cos[2\pi(2000)t]$ is sampled with an ideal sampler at a rate $f_s = 8$ kHz. to yield the DT signal $x_d[n]$. The signal $x_d[n]$ is filtered with a digital filter with frequency response $H(e^{j\omega})$ given below to yield the DT signal $y_d[n]$ .This signal is then converted to the CT signal $y_a(t)$ using an ideal D/A convertor with a cutoff filter at 4 kHz. Find $y_a(t)$.



$X_a(t) = \cos(2\pi(2000)t)$

$X_a(f) = \frac{1}{2}\left(\delta(f-2000) + \delta(f+2000)\right)$

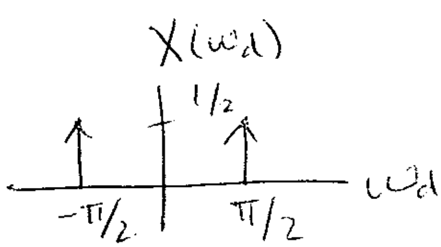


A/D

2kHz

$X(\omega) = X_a(f)\Big|_{\omega_d = \frac{2\pi f_a}{f_s}} = \frac{1}{2}\left(\delta(\omega - \frac{\pi}{2}) + \delta(\omega + \frac{\pi}{2})\right)$

8 kHz

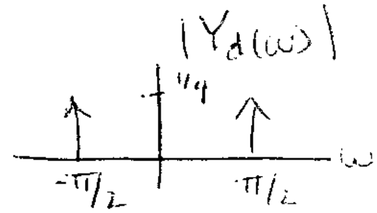


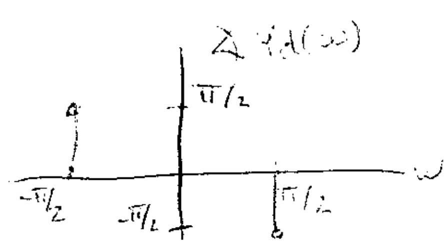
H(ω)

$|H(\omega)|\Big|_{\omega = \pm\frac{\pi}{2}} = \frac{1}{2}$

$\measuredangle H(\omega)\Big|_{\omega = \frac{\pi}{2}} = \frac{-\pi}{2}$

$\measuredangle H(\omega)\Big|_{\omega = -\frac{\pi}{2}} = +\frac{\pi}{2}$





4. (continued)

$$Y_d(\omega) = \frac{1}{4}\left[\delta(\omega-\tfrac{\pi}{2})e^{-j\pi/2} + \delta(\omega+\tfrac{\pi}{2})e^{j\pi/2}\right]$$

$$= \frac{1}{4}e^{-j\pi/2}\left(\delta(\omega-\tfrac{\pi}{2}) - \delta(\omega+\tfrac{\pi}{2})\right)$$

$$= \frac{-j}{4}\left(\delta(\omega-\tfrac{\pi}{2}) - \delta(\omega+\tfrac{\pi}{2})\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2j}\left(\delta(\omega-\tfrac{\pi}{2}) - \delta(\omega+\tfrac{\pi}{2})\right)$$

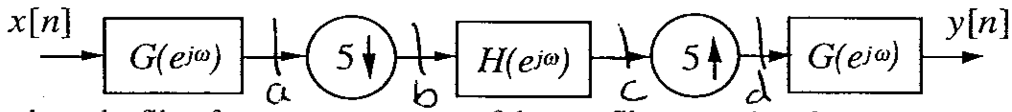
- D/A $\quad f = \dfrac{\omega f_s}{2\pi}$

$$Y_a(f) = \frac{1}{2}\cdot\frac{1}{2j}\left(\delta(f-2000) - \delta(f+2000)\right)$$

$f_c = 4\,\text{kHz} \Rightarrow$ Nothing above $\pm 2\,\text{kHz}$, so filter has no effect

- $y(t) = \frac{1}{2}\sin(2\pi(2000)t)$ //

5. (20 pts.) Consider the system shown below.



where the filter frequency responses of the two filters are given by

$$G(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/5 \\ 0, & \text{else} \end{cases} \qquad H(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

Find a simple expression for the frequency response of the overall system.

$$Y_a(\omega) = \begin{cases} X(\omega) & |\omega| < \pi/5 \\ 0 & \text{else} \end{cases}$$

$$Y_b(\omega) = \frac{1}{5}\sum_{k=0}^{4} Y_a\left(\frac{\omega+2\pi k}{5}\right)$$

$$Y_c(\omega) = \left(\frac{1}{5}\sum_{k=0}^{4} Y_a\left(\frac{\omega+2\pi k}{5}\right)\right) H(\omega)$$

$$= \frac{1}{5}\sum_{k=0}^{4} Y_a\left(\frac{\omega+2\pi k}{5}\right) e^{-j\omega/2}\cos(\omega/2)$$

$$Y_d(\omega) = Y_c(5\omega)$$

$$= \frac{1}{5}\sum_{k=0}^{4} Y_a\left(\frac{5\omega+2\pi k}{5}\right) e^{-j5\omega/2}\cos(5\omega/2)$$

$$Y(\omega) = \begin{cases} Y_d(\omega) & |\omega| < \pi/5 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{5} e^{-j5\omega/2}\cos\left(\frac{5\omega}{2}\right) X(\omega) & |\omega| < \pi/5 \\ 0 & \text{else} \end{cases}$$

5. (continued)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \begin{cases} \frac{1}{5} e^{-j\frac{5\omega}{2}} \cos\left(\frac{5\omega}{2}\right) & |\omega| < \frac{\pi}{5} \\ 0 & \text{else} \end{cases}$$

$$H(\omega) = \frac{1}{5} e^{-j5\omega/2} \cos(5\omega/2) \underbrace{\text{rect}\left(\frac{5\omega}{2\pi}\right)}_{\text{rect}\left(\frac{\omega - 0}{2\pi/5}\right)} \quad //$$