6 May 2022 **Name:** Solution

ECE 438 **Final Exam** **Spring 2022**

- You have 120 minutes to work the following **five** problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (30 pts) Consider the real-valued continuous-time signal $x(t)$ defined by

$$x(t) = \begin{cases} -1, & -1 \le t < 0 \\ 1, & 0 \le t < 1 \end{cases}$$

a. (2) Carefully sketch $x(t)$ being sure to dimension both axes.

b. (12) Find a simple expression for the CTFT $X(f)$ of $x(t)$. Your answer should not include any operators, such as convolution, rep, or comb.

c. (3) Carefully sketch $X(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

Now consider the signal $y(t) = \text{rep}_4\left[x(t)\right]$

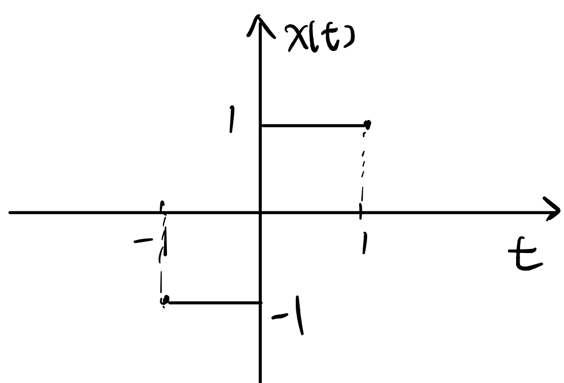
d. (2) Carefully sketch $y(t)$ being sure to dimension both axes.

e. (8) Find a simple expression for the CTFT $Y(f)$ of $y(t)$. Your answer should not include any operators, such as convolution, rep, or comb.

f. (3) Carefully sketch $Y(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

Solution:

a. 2pts



Note: $X(f) = -2j\sin\pi f \cdot \text{sinc} f$

$= -2j\pi f\,\text{sinc}^2 f$

Equation (1)

b. $x(t) = \text{rect}(t-\frac{1}{2}) - \text{rect}(t+\frac{1}{2})$ 4pts

CTFT: according to the pair $\text{rect}(t) \overset{CTFT}{\longleftrightarrow} \text{sinc}(f)$

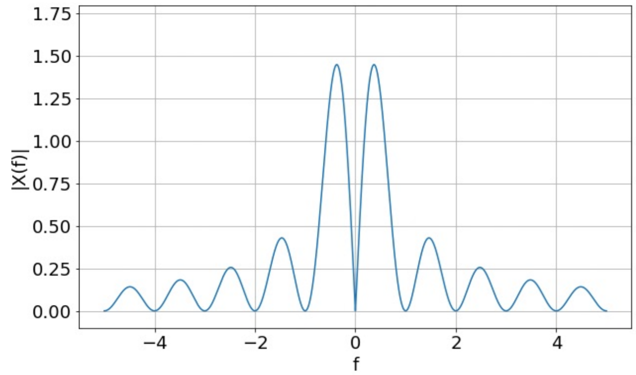
$X(f) = \text{sinc}(f)\,e^{-j2\pi f\cdot\frac{1}{2}} - \text{sinc}(f)\,e^{j2\pi f\cdot\frac{1}{2}} = \text{sinc}(f)\left(e^{-j\pi f} - e^{j\pi f}\right)$ 6pts

$= -2j\sin\pi f \cdot \text{sinc}(f)$ 2pts

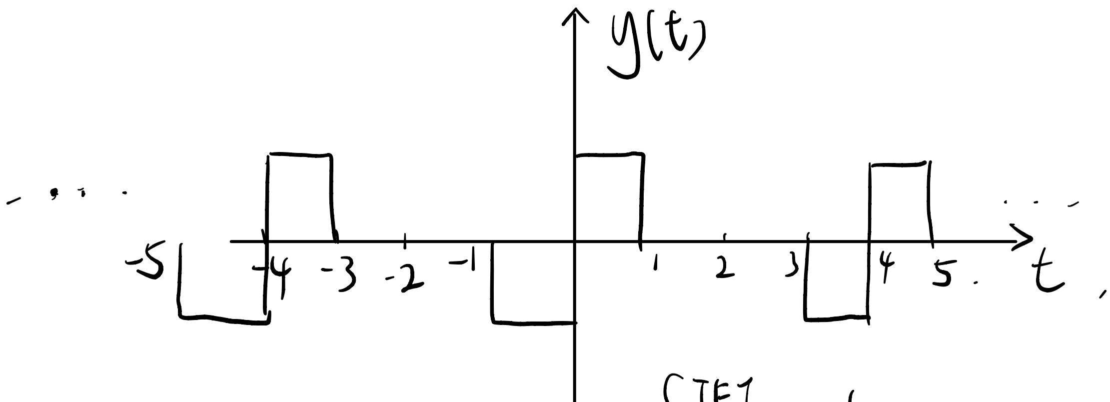
1. (continued – 1)

c. Sketch $|X(f)|$

3pts



d. $y(t) = \text{rep}_4\left[x(t)\right]$



2pts

e. We know $\text{rep}_T[x(t)] \overset{CTFT}{\Longleftrightarrow} \frac{1}{T}\text{comb}_{\frac{1}{T}}[X(f)]$.

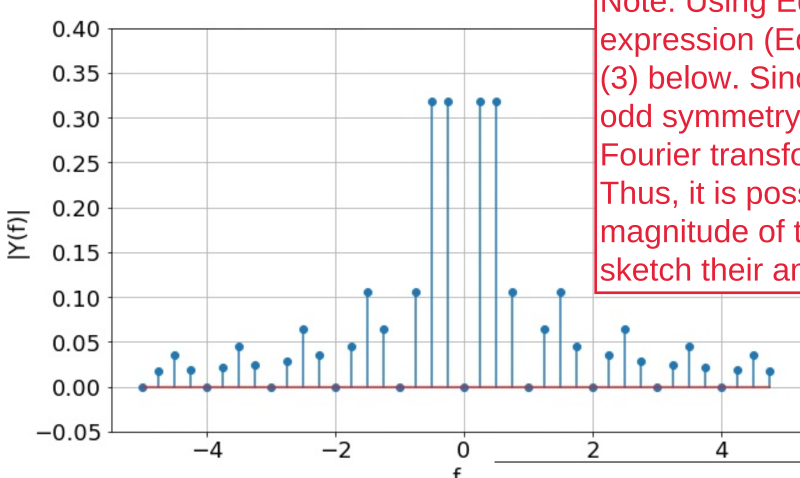
$\Rightarrow Y(f) = \frac{1}{T}\text{comb}_{\frac{1}{T}}(X(f))$ where $T=4$ 4pts

$$= \frac{1}{4}X(f)\cdot\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{1}{4}k\right) = \frac{1}{4}\left(-2j\sin\pi f\cdot\text{sinc} f\right)\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{1}{4}k\right)$$

2pts Equation (2) 2pts

f. Sketch $|Y(f)|$

3pts



Note: Using Eq. (1) above, the above expression (Eq. (2) can be written as Eq. (3) below. Since both x(t) and y(t) have odd symmetry in the time domain, their Fourier transforms are purely imaginary. Thus, it is possible to sketch either the magnitude of the Fourier transforms, or to sketch their amplitudes along the j-axis.

$$Y(f) = \frac{-j\pi}{2}\sum_{k=-\infty}^{\infty}\frac{k}{4}\text{sinc}^2\left(\frac{k}{4}\right)\delta\left(f-\frac{k}{4}\right)$$

Equation (3)

2. (30 pts.) Consider the causal DT system described by the difference equation

$$y[n] = x[n] + \tfrac{1}{3} y[n-1] \quad (*)$$

a. (10) Use Z-transform techniques, including a partial fraction expansion, to find the response $y[n]$ of this system to the input $x[n] = u[n]$.

b. (5) Find the unit sample response $h[n]$ for this system.

c. (10) Find the response $y[n]$ of this system to the input $x[n]=u[n]$ by convolving $h[n]$ with $x[n]$

d. (5) Carefully sketch the input $x[n]$=u[n] and the output $y[n]$, as functions of the independent variable n. Be sure to dimension both axes.

Solution:

a. apply Z transform to (*)

$$\Rightarrow Y(z) = X(z) + \tfrac{1}{3} Y(z) z^{-1}$$

$$Y(z)\left(1 - \tfrac{1}{3} z^{-1}\right) = X(z)$$

$$Y(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} X(z) \quad \text{(3pts)}$$

When $x[n] = u[n] \Rightarrow X(z) = \frac{1}{1 - z^{-1}}$

$$\Rightarrow Y(z) = \frac{1}{(1 - \frac{1}{3} z^{-1})(1 - z^{-1})} = \frac{a}{1 - \frac{1}{3} z^{-1}} + \frac{b}{1 - z^{-1}}$$

$$a(1 - z^{-1}) + b(1 - \tfrac{1}{3} z^{-1}) = 1 \Rightarrow \begin{cases} a + b = 1 \\ a + \frac{1}{3} b = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = \frac{3}{2} \end{cases}$$

$$\Rightarrow Y(z) = -\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{3} z^{-1}} + \frac{3}{2} \cdot \frac{1}{1 - z^{-1}} \quad \text{(3pts)}$$

$\Rightarrow$ Apply Inverse Z transform to $Y(z)$

$$\Rightarrow y[n] = -\frac{1}{2} \cdot \left(\frac{1}{3}\right)^n u[n] + \frac{3}{2} u[n]$$

$$= u[n]\left[\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n\right] \quad \text{(4pts)}$$

2. (continued – 1)

b. $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{3}z^{-1}} \Rightarrow h(n) = (\frac{1}{3})^n u[n]$ (5 pts)

c. $y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[n-m] \cdot h[m]$

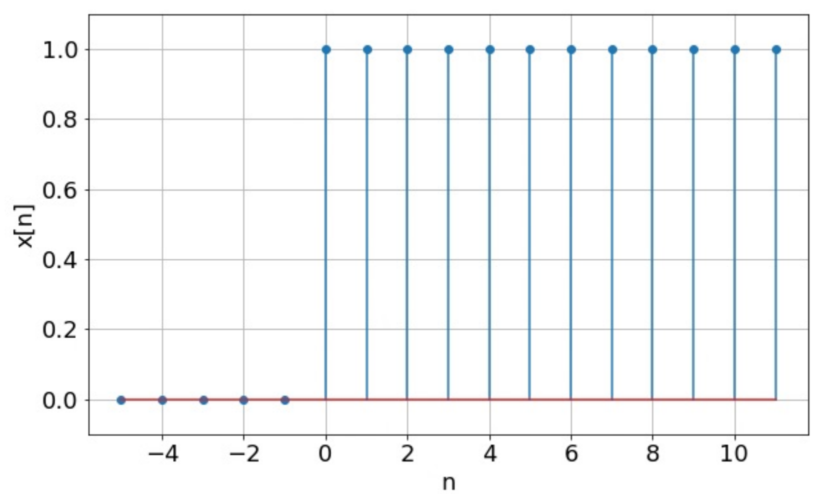
$y[n] = \sum_{m=0}^{\infty} (\frac{1}{3})^m x[n-m]$ (3 pts)

if $x[n] = u[n]$,

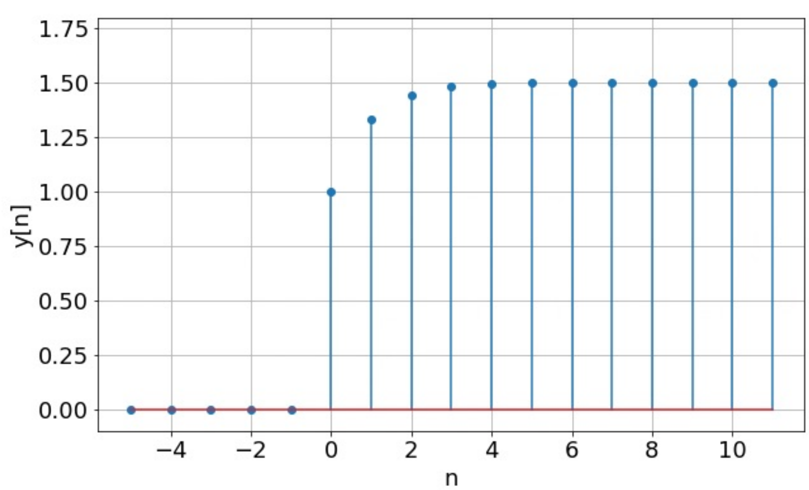
$\Rightarrow y[n] = \sum_{m=0}^{\infty} (\frac{1}{3})^m u[n-m] = u[n] \sum_{m=0}^{n} (\frac{1}{3})^m$ (3 pts)

$= u[n] \left[\frac{1-(\frac{1}{3})^{n+1}}{1-\frac{1}{3}} \right] = u[n] \left[\frac{3}{2} - \frac{1}{2}(\frac{1}{3})^n \right]$ (4 pts)

d.



(2 pts)



(3 pts)

3. (25 pts) Fast Fourier Transform Algorithm

   a. (4) Calculate the *approximate* number of complex operations (COs) required to compute a 12-point DFT by directly evaluating the 12-point DFT sum. Here a complex operation is taken to mean 1 complex addition and 1 complex multiplication.

   b. (16) Derive a complete set of equations to show how a 12-point Discrete Fourier Transform (DFT) can be calculated via decimation-in-time in terms of three stages, where the first two stages consist of 2-point DFTs and the third stage consists of 3-point DFTs

   c. (5) Based on your answer to part b) above, calculate the *approximate* number of complex operations (COs) required to compute a 12-point DFT using your FFT algorithm.

Note: you do **not** need to provide a flow diagram for your 12-point DFT algorithm; and you will **not** receive credit if you do provide one.

3a. The DFT Formula is

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

Each X_k requires N multiplications and N-1 additions, So, there are approximately N complex operations for large N. Therefore, there are

$$N * N = N^2 = 144 \text{ complex operations}$$

Rubric: Correct Number of Complex Operations +4 points

b. Based on the Forward DFT, we have

$$X^{(12)}[k] = \sum_{n=0}^{11} x[n] e^{-j2\pi \frac{kn}{12}}$$

Since 12 = 3 * 2 * 2 we construct 3 4-point FFT, decimating the input points by 3.

Stage III:

let $n = 3m + p$, $m = 0, \ldots, 3$, $p = 0, 1, 2$. Rewrite the DFT as

$$X^{(12)}[k] = \sum_{p=0}^{2} \sum_{m=0}^{3} x[3m+p] e^{-j2\pi \frac{k(3m+p)}{12}}$$

let $x_p[m] = x[3m+p]$

$$X^{(12)}[k] = \sum_{p=0}^{2} e^{-j2\pi \frac{kp}{12}} \sum_{m=0}^{3} x_p[m] e^{-j2\pi \frac{km}{4}}$$

let $$X_p^{(4)}[k] = \sum_{m=0}^{3} x_p[m] e^{-j2\pi \frac{km}{4}}$$

$$X^{(12)}[k] = \sum_{p=0}^{2} e^{-j2\pi \frac{kp}{12}} X_p^{(4)}[k] = \sum_{p=0}^{2} W_{12}^{kp} X_p^{(4)}[k] \quad \text{where } W_n = e^{-j\frac{2\pi}{n}}$$

Stage II:

Let $n = 2m + p$, $m = 0, 1$ $p = 0, 1$. Rewrite the DFT into

$$X^{(4)}[k] = \sum_{p=0}^{1} \sum_{m=0}^{1} x^{22}[2m+p]\, e^{-j2\pi \frac{k(2m+p)}{4}}$$

Let $x_p^{22}[m] = x^{22}[2m+p]$

$$X^{(4)}[k] = \sum_{p=0}^{1} e^{-j2\pi \frac{kp}{4}} \sum_{m=0}^{1} x_p^{22}[m]\, e^{-j2\pi \frac{km}{2}}$$

Let $X_p^{(2)}[k] = \sum_{m=0}^{1} x_p^{22}[m]\, e^{-j2\pi \frac{km}{2}}$

$$X^{(4)}[k] = \sum_{p=0}^{1} e^{-j2\pi \frac{kp}{4}} X_p^{(2)}[k] = \sum_{p=0}^{1} W_4^{kp} X_p^{(2)}[k] \quad \text{where } W_N = e^{-j\frac{2\pi}{N}}$$

Stage I:

We generate the 2-point FFT:

$$X^{(2)}[k] = \sum_{n=0}^{1} x^{12}[n]\, e^{-j2\pi \frac{kn}{2}}$$

$$= x^{12}[0] + x^{12}[1]\, e^{-j\pi k}$$

$$\Rightarrow \begin{cases} X^{(2)}[0] = x^{12}[0] + x^{12}[1] \\ X^{(2)}[1] = x^{12}[0] - x^{12}[1] \end{cases}$$

Rubric: Correct Forward DFT Equation : +1 Point

Correct Stage III : +5 points

Correct Stage II : +5 points

Correct Stage I : +5 points

c. Stage I 6 COs If the input is complex, it will require 12 complex additions, 6 CO if each addition counts as 1/2 CO

Stage II: 3×3 = 9 COs There are 3 4-point FFTs with 4 complex multiplications

Stage III: 12×2 = 24 COs There are 12 outputs and each require 2 complex multiplications

Total: 9 + 24+6 = 39 COs

Rubric:

Correct CO for Stage I	:	+1	Point
Correct CO for Stage II	:	+1	Point
Correct CO for Stage III	:	+1	Point
Correct total number of CO	:	+2	Points

4. (25 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

$h[m,n]$		$n=-1$	$n=0$	$n=1$
m	-1	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
	0	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$
	1	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$

a. (9) Find the output $g[m,n]$ when this filter is applied to the following 9×9 input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 9×9 set of pixels in the input image.

0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	1	1	1	0	0	0
0	0	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis ($\nu=0$), the ν axis ($\mu=0$), the $\mu=\nu$, and the $\mu=-\nu$ axes.

c. (3) Using your results from parts (a) and (b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

d. (1) Is the filter DC-preserving? Why or why not?

Problem 4

(a) 9 points

$$g[m,n] = \begin{bmatrix} 0 & 0 & 0 & -1/6 & 1/3 & -1/6 & 0 & 0 & 0 \\ 0 & 0 & -1/6 & 0 & 1/3 & 0 & -1/6 & 0 & 0 \\ 0 & -1/6 & 0 & 0 & 1/3 & 0 & 0 & -1/6 & 0 \\ -1/6 & 0 & 0 & 1/6 & 0 & 1/6 & 0 & 0 & -1/6 \\ -1/3 & 1/6 & 1/6 & 0 & 0 & 0 & 1/6 & 1/6 & -1/3 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ -1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 1/3 & -1/3 \\ -1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 1/6 & -1/6 \end{bmatrix}$$

(b) Observe that $h[m,n]$ is separable

$$h[m,n] = h_1[m]h_2[n]$$

$$\begin{bmatrix} -1/6 & 1/3 & -1/6 \\ -1/6 & 1/3 & -1/6 \\ -1/6 & 1/3 & -1/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1/6 & 1/3 & -1/6 \end{bmatrix}$$

and since

$$h_1[m] = \delta[m-1] + \delta[m] + \delta[m+1]$$

$$h_2[n] = -\frac{1}{6}\delta[n-1] + \frac{1}{3}\delta[n] - \frac{1}{6}\delta[n+1]$$

we have

$$\begin{aligned} H_1(\mu) &= e^{-j\mu} + 1 + e^{j\mu} \\ &= 1 + 2\cos\mu \\ H_2(\mu) &= -\frac{1}{6}e^{-j\nu} + \frac{1}{3} - \frac{1}{6}e^{j\nu} \\ &= \frac{1}{3} - \frac{1}{6}\left(e^{j\nu} + e^{-j\nu}\right) \\ &= \frac{1}{3}\left(1 - \cos\nu\right) \end{aligned}$$

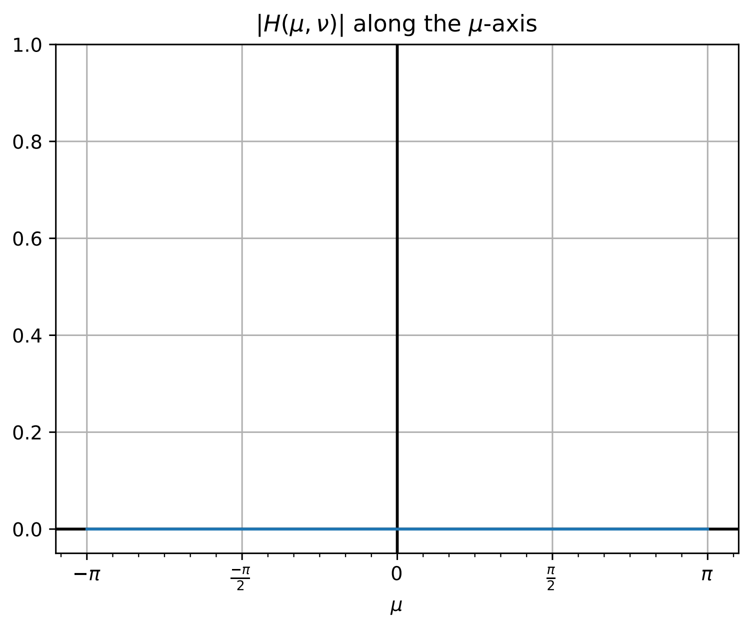
then we can calculate $H(\mu, \nu)$

$$\begin{aligned}
H(\mu,\nu) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h[m,n]e^{-j(\mu m+\nu n)} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_1[m]h_2[n]e^{-j\mu m}e^{-j\nu n} \\
&= \sum_{m=-\infty}^{\infty} h_1[m]e^{-j\mu m}\sum_{n=-\infty}^{\infty} h_2[n]e^{-j\nu n} \\
&= H_1(\mu)H_2(\nu) \\
&= \frac{1}{3}\left(1+2\cos\mu\right)\left(1-\cos\nu\right) \\
&= \frac{1}{3}+\frac{2}{3}\cos\mu-\frac{1}{3}\cos\nu-\frac{2}{3}\cos\mu\cos\nu
\end{aligned}$$

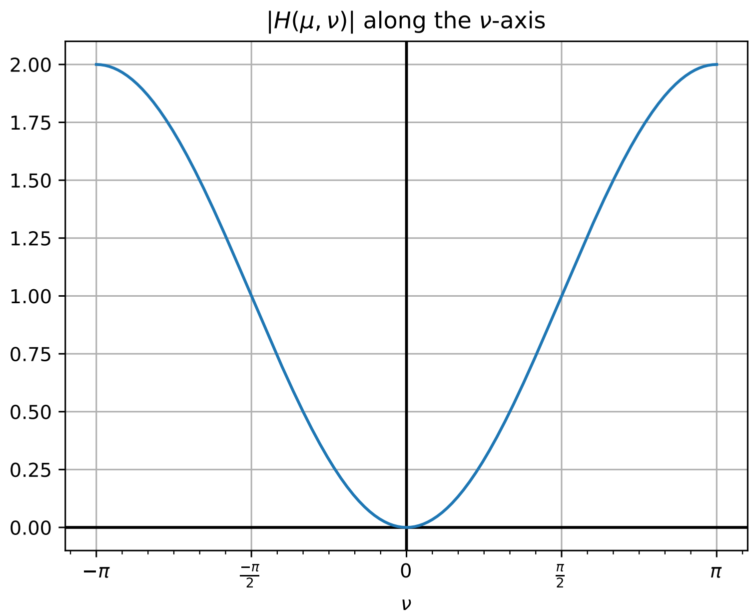
4 points

Now we sketch $|H(\mu, \nu)|$ along the μ-axis, or $|H(\mu, 0)|$, and $|H(\mu, \nu)|$ along the ν-axis, or $|H(0, \nu)|$.

$$\begin{aligned}
H(\mu,0) &= \frac{1}{3}+\frac{2}{3}\cos\mu-\frac{1}{3}-\frac{2}{3}\cos\mu \\
&= 0 \\
H(0,\nu) &= \frac{1}{3}+\frac{2}{3}-\frac{1}{3}\cos\nu-\frac{2}{3}\cos\nu \\
&= 1-\cos\nu
\end{aligned}$$



2 points

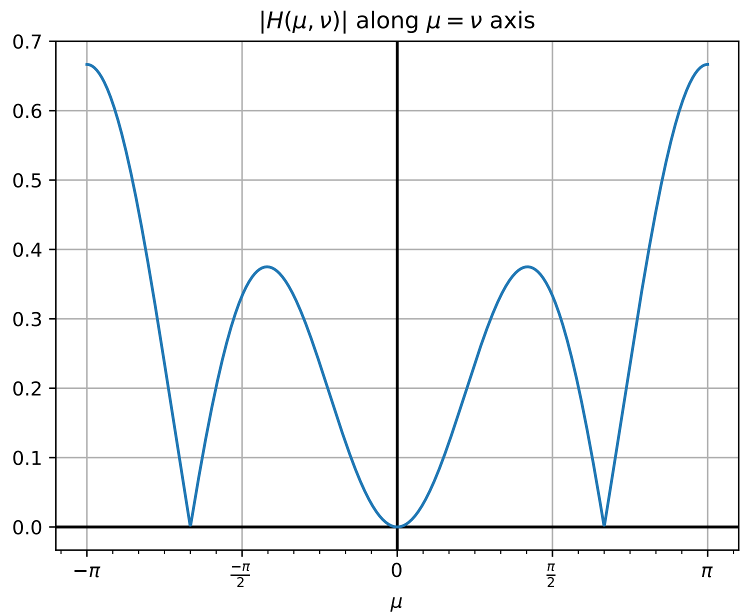


2 points

Next, since $\mu = \nu$, we have

$$\begin{aligned} H(\mu, \nu) &= H(\mu, \mu) \\ &= \frac{1}{3} + \frac{1}{3}\cos\mu - \frac{2}{3}\cos^2\mu \\ &= \frac{1}{3} + \frac{1}{3}\cos\mu - \frac{2}{3}\frac{1+\cos(2\mu)}{2} \\ &= \frac{1}{3}\cos\mu - \frac{1}{3}\cos(2\mu) \\ &= \frac{1}{3}\left(\cos\mu - \cos(2\mu)\right) \end{aligned}$$

We can now sketch $|H(\mu, \nu)|$ along the $\mu = \nu$-axis

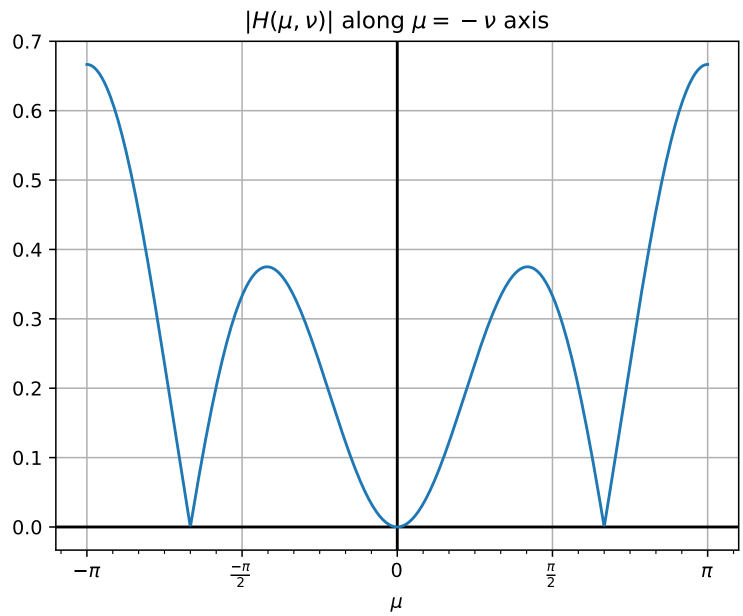


2 points

Now, since $\mu = -\nu$, we have

$$
\begin{aligned}
H(\mu, \nu) &= H(\mu, -\mu) \\
&= \frac{1}{3} + \frac{2}{3}\cos\mu - \frac{1}{3}\cos(-\mu) - \frac{2}{3}\cos\mu\cos(-\mu) \\
&= \frac{1}{3} + \frac{1}{3}\cos\mu - \frac{2}{3}\cos^2\mu, \quad \text{since } \cos(\mu) = \cos(-\mu) \\
&= \frac{1}{3}\left(\cos\mu - \cos(2\mu)\right)
\end{aligned}
$$

We can now sketch $|H(\mu, \nu)|$ along the $\mu = -\nu$ axis



2 points

3 points (c) i. The filter rejects all frequencies along the μ-axis. Notice that in the output $g[m, n]$, the bottom edge of the region of 1's is no longer visible.

ii. It is a high-pass filter along the ν-axis. Notice that in the output $g[m, n]$, the left and right edges of the region of 1's are preserved.

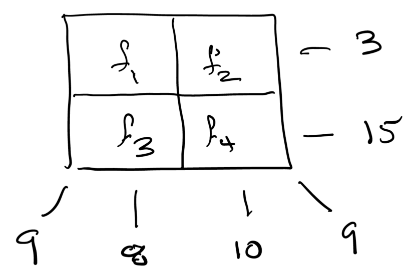
iii. Along $\mu = \nu$ or $\mu = -\nu$ axis, the filter rejects the frequencies $\mu = 0$ and $|\mu| = \frac{2}{3}\pi$. In the output $g[m, n]$, the diagonal edges are preserved.

iv. The center of the region of 1's consists of only 0's after the filter is applied, because this filter is not DC-preserving.

1 point (d) This filter is not DC-preserving.

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m, n] = 0$$

5. (15 pts.) Consider a 2×2 object with the four unknown attenuation values f_1, f_2, f_3, f_4. Suppose that the ray sums for this object are as shown below, where it is assumed that the weights $w_i, i = 1,...,4$ have value either 0 or 1.



Use the iterative algebraic reconstruction technique to determine a set of attenuation values f_1, f_2, f_3, f_4 that yield the correct values for the six ray sums.

Problem 5

Here is the pseudo-code for the iterative Algebraic Reconstruction Technique.

Algorithm 1 Algebraic Reconstruction Technique

Initialization:

$k \leftarrow 1$ ▷ k is the index of step.

$f_j^k \leftarrow 0$ ▷ $j \in 1, 2, ..., N$, which is the index of attenuation values f.

for $k = 1$ to K **do** ▷ K is the number of steps.

  for $i = 1$ to M **do** ▷ M is the number of projections in each step.

    $p_i^k \leftarrow \sum_{j=1}^{N} w_{i,j} \cdot f_j^k$

    $e_i^k \leftarrow p_i^k - P_i$ ▷ P_i is the ground truth ray sum.

    $f_j^{k+1} \leftarrow f_j^k - \frac{w_{i,j} \cdot e_i^k}{\sum_{l=1}^{N} w_{i,l}^2}$ ▷ Normalize the error and update the attenuation values.

  end for

end for

The pseudo-code is not required.

Now let's use the iterative algebraic reconstruction technique to determine the attenuation values.

Step 0, Initialize all $f = 0$, i.e., $f_1^1 = f_2^1 = f_3^1 = f_4^1 = 0$, as shown in the following figure.

0	0	——	0	3
0	0	——	0	15

Step $k = 1$, look at the above two ray sums in the horizontal direction. For the first row $i = 1$, and the second row $i = 2$. We have:

$p_1^k = \sum_{j=1}^{N} w_{1,j} \cdot f_j^k = w_{1,1} \cdot f_1^k + w_{1,2} \cdot f_2^k = 0 + 0 = 0$

$p_2^k = \sum_{j=1}^{N} w_{2,j} \cdot f_j^k = w_{2,3} \cdot f_3^k + w_{2,4} \cdot f_4^k = 0 + 0 = 0$

From the problem statement, the ground truth ray sums are $P_1 = 3, P_2 = 15$.

So the errors $e_1^k = p_1^k - P_1 = 0 - 3 = -3$, and $e_2^k = p_2^k - P_2 = 0 - 15 = -15$

Then we can update the attenuation values:

$f_1^{k+1} = f_1^k - \frac{e_1^k}{2} = 0 + 1.5 = 1.5$

$f_2^{k+1} = f_2^k - \frac{e_1^k}{2} = 0 + 1.5 = 1.5$

$f_3^{k+1} = f_3^k - \frac{e_2^k}{2} = 0 + 7.5 = 7.5$

$f_4^{k+1} = f_4^k - \frac{e_2^k}{2} = 0 + 7.5 = 7.5$

1.5	1.5
7.5	7.5
|	|
9	9
8	10

Step $k=2$, look at the above two ray sums in the vertical direction. For the first column $i=1$, and the second column $i=2$. We have:

$p_1^k = \sum_{j=1}^{N} w_{1,j} \cdot f_j^k = w_{1,1} \cdot f_1^k + w_{1,3} \cdot f_3^k = 1.5 + 7.5 = 9$

$p_2^k = \sum_{j=1}^{N} w_{2,j} \cdot f_j^k = w_{2,2} \cdot f_2^k + w_{2,4} \cdot f_4^k = 1.5 + 7.5 = 9$

From the problem statement, the ground truth ray sums are $P_1 = 8, P_2 = 10$.

So the errors $e_1^k = p_1^k - P_1 = 9 - 8 = 1$, and $e_2^k = p_2^k - P_2 = 9 - 10 = -1$

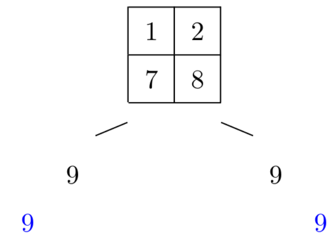
Then we can update the attenuation values:

$f_1^{k+1} = f_1^k - \frac{e_1^k}{2} = 1.5 - 0.5 = 1$

$f_2^{k+1} = f_2^k - \frac{e_2^k}{2} = 1.5 + 0.5 = 2$

$f_3^{k+1} = f_3^k - \frac{e_1^k}{2} = 7.5 - 0.5 = 7$

$f_4^{k+1} = f_4^k - \frac{e_2^k}{2} = 7.5 + 0.5 = 8$



Step $k=3$, look at the above two ray sums in the diagonal direction. For the southwest diagonal $i=1$, and the southeast diagonal $i=2$. We have:

$p_1^k = \sum_{j=1}^{N} w_{1,j} \cdot f_j^k = w_{1,2} \cdot f_2^k + w_{1,3} \cdot f_3^k = 2 + 7 = 9$

$p_2^k = \sum_{j=1}^{N} w_{2,j} \cdot f_j^k = w_{2,1} \cdot f_1^k + w_{2,4} \cdot f_4^k = 1 + 8 = 9$

From the problem statement, the ground truth ray sums are $P_1 = 9, P_2 = 9$.

So the errors $e_1^k = p_1^k - P_1 = 9 - 9 = 0$, and $e_2^k = p_2^k - P_2 = 9 - 9 = 0$

In this step, the errors are all zero, so when we update the attenuation values, the values remain unchanged:

$f_1^{k+1} = f_1^k - \frac{e_2^k}{2} = 1 + 0 = 1$

$f_2^{k+1} = f_2^k - \frac{e_1^k}{2} = 2 + 0 = 2$

$f_3^{k+1} = f_3^k - \frac{e_1^k}{2} = 7 + 0 = 7$

$f_4^{k+1} = f_4^k - \frac{e_2^k}{2} = 8 + 0 = 8$

Therefore, we determine the final attenuation values f_1, f_2, f_3, f_4 as below:

1	2
7	8

Describe how to estimate projection $p_i^k \leftarrow \sum_{j=1}^{N} w_{i,j} \cdot f_j^k$ (4pts)

Describe how to calculate error $e_i^k \leftarrow p_i^k - P_i$ (4pts)

Describe how to update attenuation values $f_j^{k+1} \leftarrow f_j^k - \frac{w_{i,j} \cdot e_i^k}{\sum_{l=1}^{N} w_{i,l}^2}$ (4pts)

Correct numerical answer, each step $k = 1, 2, 3$ (1pt)