20 April 2022 **Name:** Solution

ECE 438 **Exam No. 3** **Spring 2022**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (20 pts.) Consider two random variables X and Y which are jointly distributed according to the following bivariate density function

$$f_{XY}(x,y)=\begin{cases} 4xy, & 0\le x\le 1 \text{ and } 0\le y\le 1 \\ 0, & \text{else} \end{cases}.$$

   a. (5) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

   b. (2) Are X and Y independent?

   c. (13) Find the mean and variance of X and Y and the correlation coefficient ρ_{XY} between them.

1. $f_{XY}(x,y) = \begin{cases} 4xy & ; \quad 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & ; \quad \text{Else} \end{cases}$

a. Find the marginal densities $f_X(x)$ and $f_Y(y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^1 4xy\,dy$$

$$= 4x\left(\frac{y^2}{2}\right)\Big|_0^1$$

$$= 2x \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^1 4xy\,dx$$

$$= 4y\left(\frac{x^2}{2}\right)\Big|_0^1$$

$$= 2y \qquad 0 \le y \le 1$$

Rubric:

Correct Equation for computing $f_X(x)$	+1 point
Correct Equation for Computing $f_Y(y)$	+1 point
Correct Marginal Density $f_X(x)$	+1.5 points
Correct Marginal Density $f_Y(y)$	+1.5 points

b. Are X and Y independent?

X and Y are independent if

$f_X(x) f_Y(y) = f_{XY}(x,y)$

$f_X(x) f_Y(y) = (2x)(2y)$

$= 4xy$

$f_{XY}(x,y) = 4xy$

$4xy = 4xy$

$\therefore f_X(x) f_Y(y) = f_{XY}(x,y)$

Since $f_X(x) f_Y(y) = f_{XY}(x,y)$, X and Y are independent

Rubric: Equation for Independence +1 point

X and Y are Independent +1 point

c. Find the mean and variance of X and Y and the correlation coefficient ρ_{XY} between them

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^1 x(2x)\,dx$$

$$= 2\left(\frac{x^3}{3}\right)\Bigg|_0^1$$

$$= \frac{2}{3}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^1 x^2(2x)\,dx$$

$$= 2\left(\frac{x^4}{4}\right|_0^1$$

$$= \frac{1}{2}$$

$$Var[X] = E[X^2] - E[X]^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{18}$$

Similarly, calculate the mean and variance of Y

$$E[Y] = \frac{2}{3}$$

$$Var[Y] = \frac{1}{18}$$

Calculating the correlation coefficient between them

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy(4xy)\,dx\,dy$$

$$= \frac{4}{9}$$

$$\rho_{xy} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}}$$

$$= \frac{\frac{4}{9} - \frac{4}{9}}{\frac{1}{18}}$$

$$= 0$$

Rubric	Correct Mean of X	+ 2 points
	Correct Variance of X	+ 2 points
	Correct Mean of Y	+ 2 points
	Correct Variance of Y	+ 2 points
	Correct Evaluation of E[XY]	+ 2 points
	Correct correlation coefficient ρ_{xy}	+ 3 points

2. (30 pts) Continuous-time speech model and spectral analysis.

Consider a CT vocal tract model for a fixed phoneme with just one formant at 400 Hz with CT impulse response $v(t)$ given by

$$v(t)=\cos(2\pi(400)t)\mathrm{rect}\left(\frac{t-2.5\times10^{-3}}{5\times10^{-3}}\right).$$

Consider that the vocal tract is driven by an excitation waveform $e(t)$ consisting of ideal impulses separated by the pitch period. Assume that the pitch frequency is 100 Hz.

a. (5) Carefully sketch the vocal tract response $v(t)$. Be sure to dimension all important quantities along the time axis.

b. (4) Carefully sketch the CT speech waveform $s(t)$. Be sure to dimension all important quantities along the time axis.

c. (7) Find an expression for the CTFT $V(f)$ of the vocal tract response $v(t)$. Your expression should not contain any operators.

e. (3) Carefully sketch $V(f)$. Be sure to dimension all important quantities along the frequency axis. In your sketch, you may ignore any phase factors.

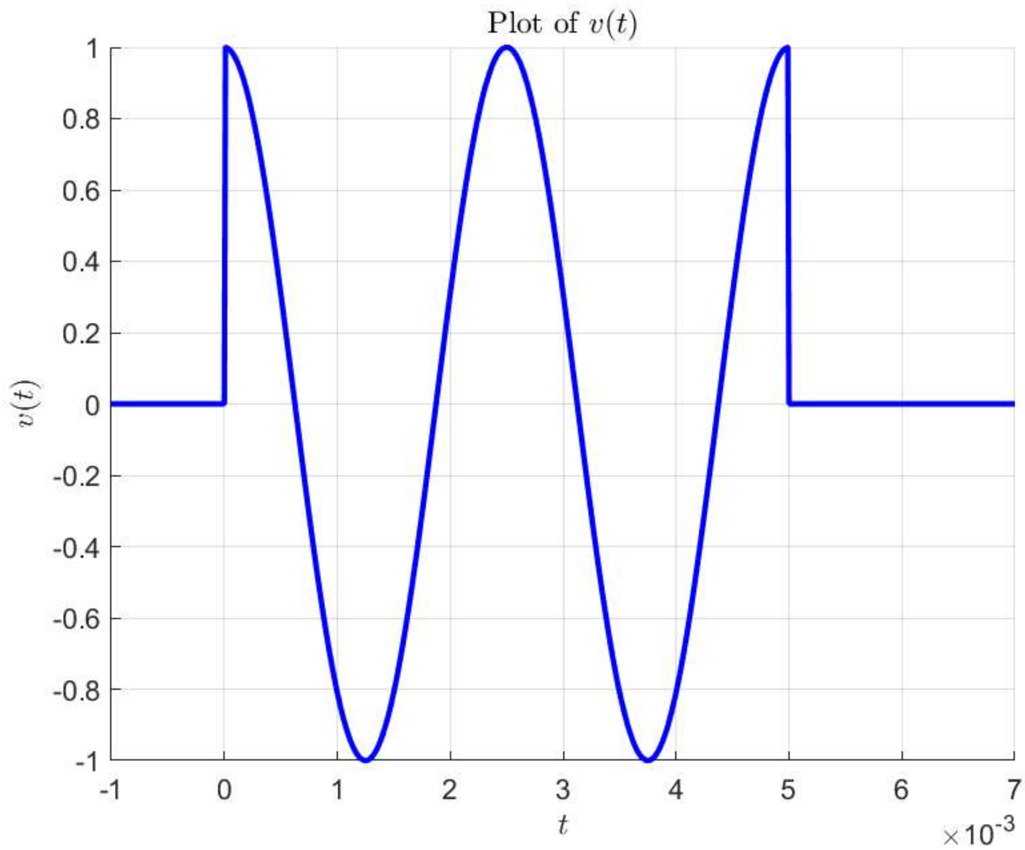
d. (8) Find an expression for the CTFT $S(f)$ of the speech waveform $s(t)$. Your expression should not contain any operators.

f. (3) Carefully sketch $S(f)$. Be sure to dimension all important quantities along the frequency axis. In your sketch, you may ignore any phase factors.

Problem 2

a.

The plot of $v(t)$ is :



Abscissa unit (1pt)

Ordinate unit (1pt)

Correct shape. (1pt)

Rect window length 5ms (1pt)

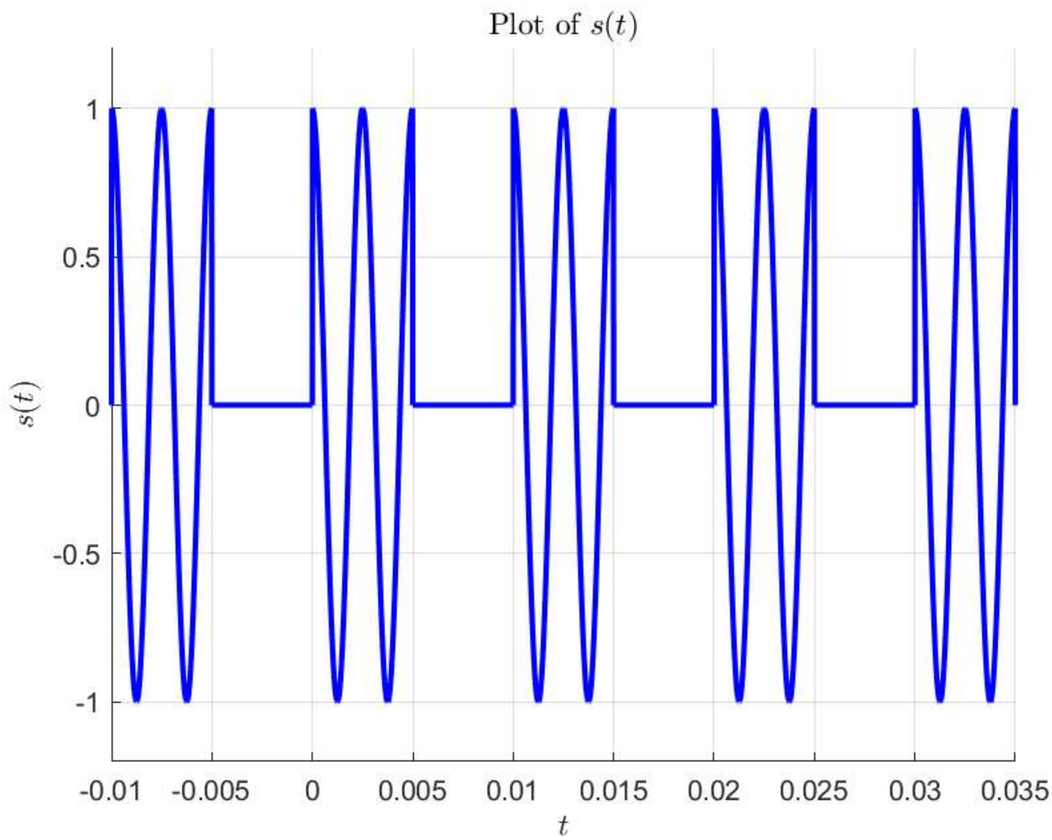
Period 2.5ms (1pt)

b.

The pitch frequency is 100 Hz, so the pitch period is $P = 0.01$ s. Thus we have that:

$$\begin{aligned} s(t) &= \text{rep}_P\{v(t)\} \\ &= \sum_{k=-\infty}^{\infty} v(t - 0.01k), \;\; k \in \mathbb{Z} \end{aligned}$$

The plot of $s(t)$ is:



Abscissa unit (1pt)

Ordinate unit (1pt)

Correct Rep pattern. (1pt)

Rep period 0.01s (1pt)

c.

$$v(t) = \cos(2\pi(400)t) \cdot \text{rect}\left(\frac{t - 2.5 \times 10^{-3}}{5 \times 10^{-3}}\right)$$

$$\text{CTFT}\{v(t)\} = \text{CTFT}\{\cos(2\pi(400)t)\} * \text{CTFT}\left\{\text{rect}\left(\frac{t - 2.5 \times 10^{-3}}{5 \times 10^{-3}}\right)\right\} \quad \text{(convolution } 1pt)$$

For simplicity, let's denote the above equation as:

$$V(f) = V_1(f) * V_2(f)$$

Use the CTFT pairs, we have that:

$$V_1(f) = \text{CTFT}\{\cos(2\pi(400)t)\}$$
$$= \frac{1}{2}\{\delta(f-400)+\delta(f+400)\} \quad \color{red}{(2pts)}$$

and

$$V_2(f) = \text{CTFT}\left\{\text{rect}\left(\frac{t-2.5\times10^{-3}}{5\times10^{-3}}\right)\right\}$$
$$= 5\times10^{-3}\cdot\text{sinc}\left\{5\times10^{-3}f\right\}\cdot e^{-j2\pi f\cdot2.5\times10^{-3}} \quad \color{red}{(2pts)}$$

Hence the CTFT of $v(t)$ is:

$$V(f) = V_1(f) * V_2(f)$$
$$= \frac{1}{2}\{\delta(f-400)+\delta(f+400)\} * V_2(f)$$
$$= \frac{1}{2}\{V_2(f-400)+V_2(f+400)\}$$
$$= 2.5\times10^{-3}\begin{pmatrix}\text{sinc}\left\{5\times10^{-3}(f-400)\right\}\cdot e^{-j2\pi(f-400)\cdot2.5\times10^{-3}}\\ +\text{sinc}\left\{5\times10^{-3}(f+400)\right\}\cdot e^{-j2\pi(f+400)\cdot2.5\times10^{-3}}\end{pmatrix}$$

$\color{red}{(2pts)}$

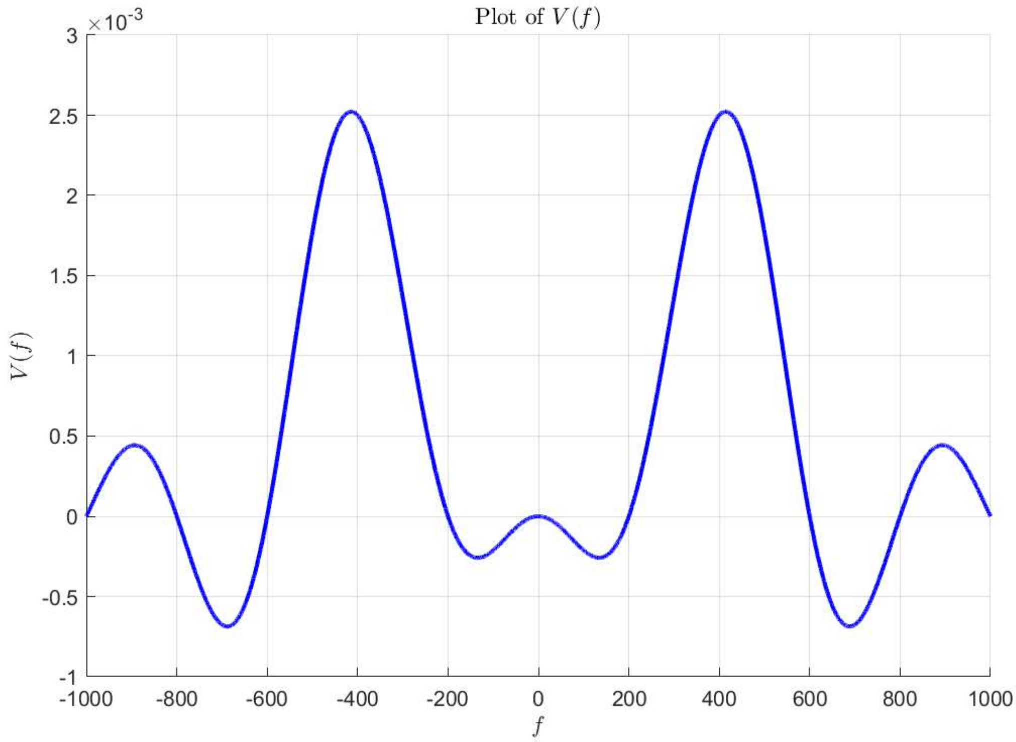
d.

From prob b we know that $s(t) = \text{rep}_{0.01}\{v(t)\}$ So the CTFT of $s(t)$ is:

$$S(f) = \text{CTFT}\{s(t)\}$$
$$= \text{CTFT}\{\text{rep}_{0.01}\{v(t)\}\} \quad \color{red}{(1pt)}$$
$$= 100\ \text{comb}_{100}\{V(f)\} \quad \color{red}{(2pts)}$$
$$= 100\sum_{k=-\infty}^{\infty} V(100k)\ \delta(f-100k) \quad \color{red}{(2pts)}$$
$$= 0.25\sum_{k=-\infty}^{\infty}\begin{pmatrix}\text{sinc}\left\{5\times10^{-3}(100k-400)\right\}\cdot e^{-j2\pi(100k-400)\cdot2.5\times10^{-3}}\\ +\text{sinc}\left\{5\times10^{-3}(100k+400)\right\}\cdot e^{-j2\pi(100k+400)\cdot2.5\times10^{-3}}\end{pmatrix}\delta(f-100k) \quad \color{red}{(3pts)}$$
$$= 0.25\sum_{k=-\infty}^{\infty}\left(\text{sinc}\left\{\tfrac{k-4}{2}\right\}\cdot e^{-j2\pi\frac{k-4}{4}}+\text{sinc}\left\{\tfrac{k+4}{2}\right\}\cdot e^{-j2\pi\frac{k+4}{4}}\right)\delta(f-100k)$$
$$= 0.25\sum_{k=-\infty}^{\infty}\left(\text{sinc}\left\{\tfrac{k-4}{2}\right\}\cdot e^{-j2\pi\frac{k}{4}}+\text{sinc}\left\{\tfrac{k+4}{2}\right\}\cdot e^{-j2\pi\frac{k}{4}}\right)\delta(f-100k)$$
$$= 0.25\sum_{k=-\infty}^{\infty}\left(\text{sinc}\left\{\tfrac{k-4}{2}\right\}+\text{sinc}\left\{\tfrac{k+4}{2}\right\}\right)\cdot e^{-j2\pi\frac{k}{4}}\cdot\delta(f-100k)$$

e.

Here's the plot of $V(f)$, ignoring the phase factors.



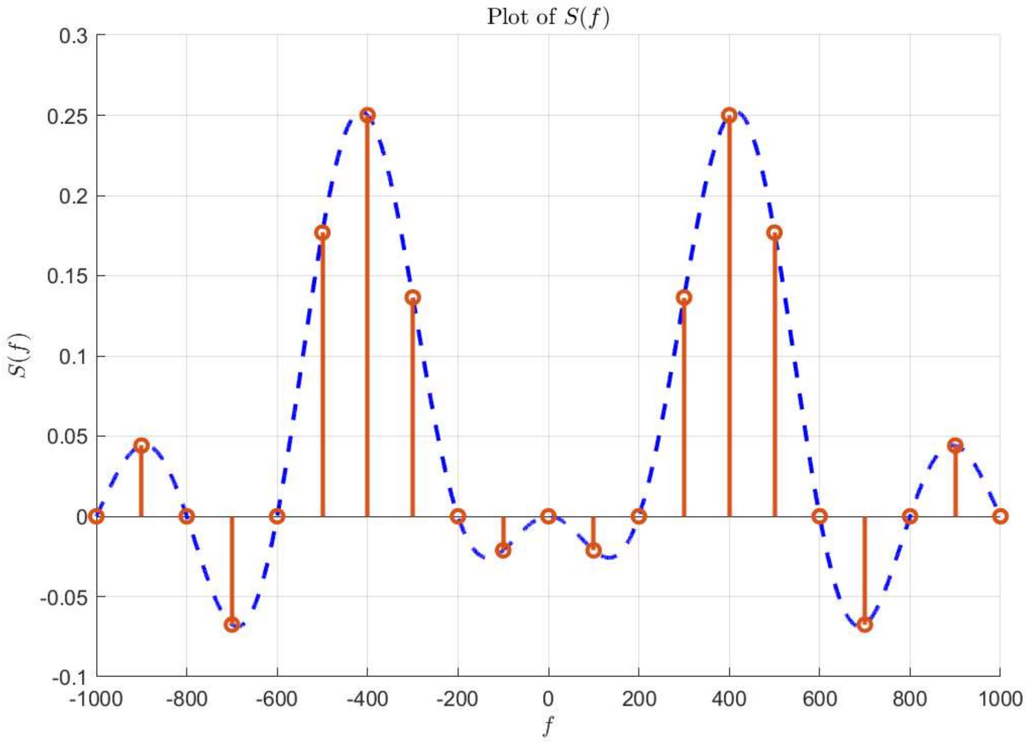
Abscissa peak at $f = 400$ and $f = -400$ (1pt)

Ordinate peak at 2.5×10^{-3} (1pt)

Correct *sinc* shape. (1pt)

f.

Because $S(f) = 100 \ \mathrm{comb}_{100}\left\{V(f)\right\}$, we can plot $S(f)$ below:



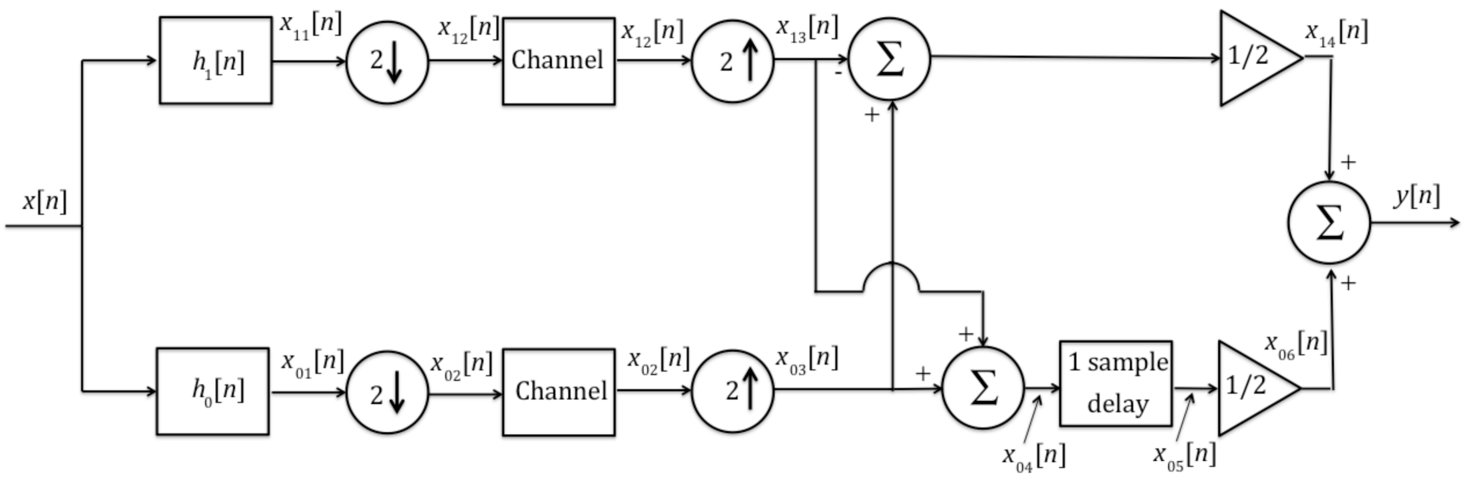
Ordinate peak at 0.25 (1pt)

Sketch stems. (1pt)

Stem occurs every $f = 100$. (1pt)

3. (30 pts) Two-channel filter bank and Haar basis wavelet.

Consider the two-channel filter bank shown below.



Here, the two filters are described by the following two equations:

$$\text{Filter 0 } (h_0[n])\text{: } x_{01}[n] = x[n] + x[n-1] \quad ,$$

$$\text{Filter 1 } (h_1[n])\text{: } x_{11}[n] = x[n] - x[n-1] \, .$$

The 1-sample delay is described by:

$$x_{05}[n] = x_{04}[n-1] \, .$$

Note that the Channel inputs and outputs are identical. So the channel does nothing to the signals.

Suppose that the input to this system is given by

n	...	-1	0	1	2	3	4	...
$x[n]$	...	0	1	2	3	0	0	...

a. {27) Tabulate the output signals $x_{01}[n]$, $x_{11}[n]$, $x_{02}[n]$, $x_{12}[n]$, $x_{03}[n]$, $x_{13}[n]$, $x_{04}[n]$, $x_{14}[n]$, $x_{05}[n]$, $x_{06}[n]$, and $y[n]$.

b. (3) Comment on the relation between $y[n]$ and $x[n]$, and what this system is doing.

Problem 3

(a)

n	-1	0	1	2	3	4	
$x[n]$	0	1	2	3	0	0	
$x_{01}[n]$	0	1	3	5	3	0	3 pts
$x_{11}[n]$	0	1	1	1	-3	0	3 pts
$x_{02}[n]$	0	1	5	0	0	0	3 pts
$x_{12}[n]$	0	1	1	0	0	0	3 pts
$x_{03}[n]$	0	1	0	5	0	0	3 pts
$x_{13}[n]$	0	1	0	1	0	0	3 pts
$x_{04}[n]$	0	2	0	6	0	0	3 pts
$x_{14}[n]$	0	0	0	2	0	0	2 pts
$x_{06}[n]$	0	0	1	0	3	0	2 pts
$y[n]$	0	0	1	2	3	0	2 pts

(b) $y[n] = x[n-1]$ 2 pts

the system reconstructs the signal

See the following page for some additional thoughts.

but with 1-sample delay. 1 pt

3. (continued - 1)

The filters h_0[n] and h_1[n] separate the signal into a low frequency band and a high frequency band respectively. Each channel can be down-sampled by 2x with no loss of information, since, as demonstrated, recovery of the original signal with a 1-sample delay is possible. By forming a hierarchy of the front end of this system (prior to the portion labeled as "channel"), a full wavelet, multiresolution decomposition of the signal is possible.

4. (20 pts.) Adrian is concerned about his grade in ECE 438. He received grades of 40, 70, and 60, respectively, on the first three hour exams, and would like to predict what will be his likely grade on the final exam. He chooses a predictor of the following form:

$$\hat{S}(E) = aE + b,$$

where $E = 1,2,3,4$ denotes the number of the exam, $\hat{S}(E)$ is the predicted score on Exam E, and a, b are constants.

(20) Determine two simple linear equations in the unknowns a and b that when solved will yield values for a and b that minimize the mean-squared error ε between the data and this model:

$$\varepsilon = \frac{1}{3}\sum_{E=1}^{3}\left[\hat{S}(E) - S(E)\right]^2. \quad (*)$$

Here $S(E)$ denotes the data, i.e. $S(1) = 40$, $S(2) = 70$, and $S(3) = 60$. Note that you do **not** need to actually solve these two equations.

Solution: $\hat{S}(E) = aE + b$, take it into $(*)$

$$\Rightarrow \varepsilon = \frac{1}{3}\sum_{E=1}^{3}\left[aE + b - S(E)\right]^2$$

$$= \frac{1}{3}\left[(a+b-S(1))^2 + (2a+b-S(2))^2 + (3a+b-S(3))^2\right]$$

$$= \frac{1}{3}\left[(a+b-40)^2 + (2a+b-70)^2 + (3a+b-60)^2\right]$$ 4pts

$$\frac{\partial \varepsilon}{\partial a} = \frac{1}{3}\left[2(a+b-40) + 2(2a+b-70)\cdot 2 + 2(3a+b-60)\cdot 3\right]$$ 5pts

$$= \frac{2}{3}\left[a+b-40+4a+2b-140+9a+3b-180\right]$$

$$= \frac{2}{3}\left[14a+6b-360\right] = \frac{4}{3}\left[7a+3b-180\right] \quad (1)$$

$$\frac{\partial \varepsilon}{\partial b} = \frac{1}{3}\left[2(a+b-40) + 2(2a+b-70) + 2(3a+b-60)\right]$$ 5pts

$$= \frac{2}{3}\left[6a+3b-170\right] \quad (2)$$

Let $\frac{\partial \varepsilon}{\partial a} = 0$, $\frac{\partial \varepsilon}{\partial b} = 0 \Rightarrow$

(1): $7a+3b-180 = 0$ 3pts

(2): $6a+3b-170 = 0$ 3pts

$$\Rightarrow a = 10, \quad b = \frac{110}{3}$$