23 March 2022 **Name:___________________**

ECE 438 **Exam No. 2** **Spring 2022**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is cloosed book and closed notes.
- Calculators, smart phones, and smart watches are not permitted, and must be put away.

1. (25 pts) Consider the causal DT system with the following transfer function

$$H_{ZT}(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{1}{4}z^{-2}\right)}$$

a. (6) Plot the poles and zeros in the complex z plane.

b. (4) State the region of convergence for this transfer function.

c. (15) Let $H_{Freq}(\omega) = H_{ZT}\left(e^{j\omega}\right)$ be the frequency response of this system. Use the graphical approach to determine $\left|H_{Freq}\left(\frac{\pi}{6}\right)\right|$. Be sure to clearly show how you obtained your answer. It may be helpful to know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

1.

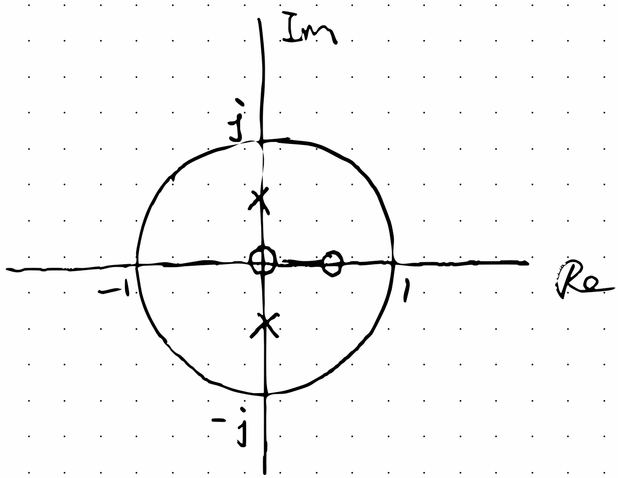
(a) $$H_{ZT}(z) = \frac{(1-\frac{1}{2}z^{-1})}{(1+\frac{1}{4}z^{-2})}$$

$$= \frac{z^2 - \frac{1}{2}z}{z^2 + \frac{1}{4}}$$

$$= \frac{z(z-\frac{1}{2})}{(z+\frac{1}{2}j)(z-\frac{1}{2}j)}$$ 2 pts

zeros: $z = 0, \frac{1}{2}$ 2 pts

poles: $z = \pm\frac{1}{2}j$ 2 pts



(b) causal system

$|z| > 1/2$ 4 pts

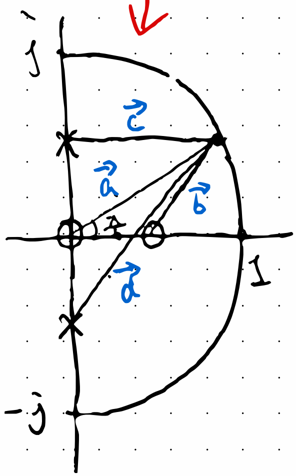
(C)

$$|H(\tfrac{\pi}{6})| = \frac{|\vec{a}|\,|\vec{b}|}{|\vec{c}|\,|\vec{d}|}$$ 2pts

4 pts

$\vec{a}$: 1 pt $\vec{c}$: 1 pt

$\vec{b}$: 1 pt $\vec{d}$: 1 pt



$$|\vec{a}| = 1$$ 2pts

$$|\vec{b}| = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2} - \tfrac{1}{2})^2}$$

$$= \sqrt{\frac{5-2\sqrt{3}}{4}}$$

$$= \frac{\sqrt{5-2\sqrt{3}}}{2}$$ 2pts

$$|\vec{c}| = \cos(\tfrac{\pi}{6}) = \frac{\sqrt{3}}{2}$$ 2pts

$$|\vec{d}| = \sqrt{1^2 + (\tfrac{\sqrt{3}}{2})^2}$$

$$= \frac{\sqrt{7}}{2}$$ 2pts

$$|H(\tfrac{\pi}{6})| = \frac{1 \cdot \frac{\sqrt{5-2\sqrt{3}}}{2}}{\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{7}}{2}}$$

$$= \frac{\sqrt{5-2\sqrt{3}}}{\sqrt{21}} \cdot 2$$

$$= \frac{2\sqrt{21(5-2\sqrt{3})}}{21}$$ 1 pt

2. (25 pts) Fast Fourier Transform Algorithm

a. (3) Calculate the *approximate* number of complex operations (COs) required to compute a 6-point DFT by directly evaluating the 6-point DFT sum. Here a complex operation is taken to mean 1 complex addition and 1 complex multiplication.

b. (12) Derive a complete set of equations to show how a 6-point Discrete Fourier Transform (DFT) can be calculated in terms of three 2-point DFTs via decimation-in-time.

c. (5) Based on your answer to part b) above, list the complete ordering of the 6-point input to your 6-point FFT algorithm.

d. (5) Based on your answer to part b) above, calculate the *approximate* number of complex operations (COs) required to compute a 6-point DFT using your FFT algorithm.

Note: you do **not** need to provide a flow diagram for your 6-point DFT algorithm; and you will **not** receive credit if you do provide one.

2. a The DFT formula is

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

Each X_k requires N multiplications and N-1 Additions. So, there are approximately N Complex operations for large N

Therefore, there are

$$N \times N = N^2 = 36 \text{ Complex operation}$$

b. $$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi kn}{N}}$$

$$X^{(6)}[k] = \sum_{n=0}^{5} e^{-j\frac{2\pi kn}{6}}$$

Split into three 2-pt DFTs

$$X^{(6)}[k] = \sum_{n=0}^{5} x[n] e^{-j\frac{2\pi kn}{6}}$$

$$= \sum_{l=0}^{2} \sum_{n=0}^{1} x[3n+l] e^{-j\frac{2\pi k(3n+l)}{6}}$$

$$= \sum_{l=0}^{2} e^{-j\frac{2\pi kl}{6}} \sum_{n=0}^{1} x[3n+l] e^{-j\frac{2\pi kn}{2}}$$

$$= \sum_{l=0}^{2} W_6^{lk} X_l^{(2)}[k]$$

where

$$W_6^{lk} = e^{-j\frac{2\pi kl}{6}}$$

$$X_l^{(2)}[k] = \sum_{n=0}^{1} x[3n+l] e^{-j\frac{2\pi kn}{2}}$$

c.

l	n	$X_l[n]$	$3n+l$	$X[3n+l]$
0	0	$X_0[0]$	0	$X[0]$
0	1	$X_0[1]$	3	$X[3]$
1	0	$X_1[0]$	1	$X[1]$
1	1	$X_1[1]$	4	$X[4]$
2	0	$X_{22}[0]$	2	$X[2]$
0	1	$X_2[1]$	5	$X[5]$

d. Three 2-pt DFTs : $3(2^2) = 3(4) = 12$ COs

Two Complex Operations to Combine : $2(6) = 12$ COs

Total Complex Operations $= 12+12 = 24$ COs

Rubric

Part a: Correct Number of CO's : +3 points

Part b : Correct DFT Equation : +3 points

Correct Split of 6-pt DFT in terms of 3 2-pt DFT : +6 points

Correct final Answer + 3 points

Part c: Correct Ordering : +5 points

Part d: Correct Number of CO's for 2-pt DFTs : + 2 points

Correct Number of CO's to Combine : +2 points

Correct Total Number of CO's : +1 point

3. (25 pts) Spectral analysis via the DFT

Consider the 64-point signal $x[n]=\cos\left(\frac{2\pi(7)}{64}n\right), n=0,...,63$.

a. (7) Determine an exact expression for the 64-point discrete Fourier transform (DFT) $X^{(64)}\left[k\right], k=0,...,63$ of this signal.

b. (2) Are there any leakage or picket fence effects in this case? Why or why not?

Now consider the 64-point signal $x[n]=\cos\left(\frac{2\pi(15)}{128}n\right), n=0,...,63$.

c. (12) Determine an exact expression for the 64-point discrete Fourier transform (DFT) $X^{(64)}\left[k\right], k=0,...,63$ of this signal in terms of the function

$$\text{psinc}_N(\omega)=\frac{\sin(\omega N/2)}{\sin(\omega/2)}.$$

d. (4) Are there any leakage or picket fence effects in this case? Why or why not?

3. (continued - 1)

[6 Pts] (a)

$$x[n] = \cos\left(\frac{2\pi(7)}{64}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi(7)}{64}n} + e^{-j\frac{2\pi(7)}{64}n}\right)$$

By checking the transform pair,

we have $\cos(2\pi k_0 n/N) \overset{DFT}{\longleftrightarrow} \frac{N}{2}\{\delta[k-k_0]+\delta[k-(N-k_0)]\}, \ 0 \le k \le N-1$

3pt

in our case $N=64$, $k_0=7$

thus $X^{(64)}[k] = \frac{64}{2}\{\delta[k-7]+\delta[k-(64-7)]\}$

3pt $= 32\{\delta[k-7]+\delta[k-57]\}$, $k=0,1,\ldots,63$

[5 Pts] (b)

The picket fence effect is caused by sampling in the frequence domain. For a sinusoidal signal, it appears if the frequency of the sinusoid, ω_0, is not an integer multiple of $\frac{2\pi}{N}$

In (a), $\omega_0 = \frac{2\pi}{64}\cdot 7$, 7 is integer $\Rightarrow$ No picket fence effects

1.5pt 1.5pt

Truncation causes leakage. However, because of the points at which we sample in the frequency domain, we do not see it $\Rightarrow$ No leakage

2pt

3. (continued - 2)

[9 Pts] (c) $x[n] = \cos\left(\frac{2\pi(15)}{128}n\right) = \frac{1}{2}\left(\underbrace{e^{j\frac{2\pi(15)}{128}n}}_{A} + \underbrace{e^{-j\frac{2\pi(15)}{128}n}}_{B}\right)$

By checking the transform pairs,

we have $e^{j\omega_0 n} \overset{DFT}{\longleftrightarrow} \text{psinc}_N(2\pi k/N - \omega_0)\, e^{-j(2\pi k/N - \omega_0)(N-1)/2}$ 3pt

For part A: let $\omega_A = \frac{2\pi(15)}{128} = \frac{15\pi}{64}$

For part B: let $\omega_B = -\frac{2\pi(15)}{128} = -\frac{15\pi}{64}$

3pt, correct ω_A, ω_B (ω_B')

then $x[n] \overset{DFT}{\longleftrightarrow} \frac{1}{2}\Big\{ \text{psinc}_{64}\left(\frac{2\pi k}{64} - \omega_A\right) e^{-j\left(\frac{2\pi k}{64} - \omega_A\right)\frac{63}{2}}$

$+ \text{psinc}_{64}\left(\frac{2\pi k}{64} - \omega_B\right) e^{-j\left(\frac{2\pi k}{64} - \omega_B\right)\frac{63}{2}} \Big\}$

$\Rightarrow X^{(64)}[k] = \frac{1}{2}\Big\{ \text{psinc}_{64}\left(\frac{2\pi k - 15\pi}{64}\right) e^{-j\left(\frac{2\pi k - 15\pi}{64}\right)\cdot\frac{63}{2}}$

$+ \text{psinc}_{64}\left(\frac{2\pi k + 15\pi}{64}\right) e^{-j\left(\frac{2\pi k + 15\pi}{64}\right)\cdot\frac{63}{2}} \Big\}$, $k = 0, \ldots, 63$

Note: $2\pi k - 15\pi = 2\pi(k - 7.5)$; $2\pi k + 15\pi = 2\pi(k + 7.5)$

$\omega_B' = \omega_B + 2\pi = \frac{128\pi}{64} - \frac{15\pi}{64} = \frac{113\pi}{64}$

$\Rightarrow X^{(64)}[k] = \frac{1}{2}\Big\{ \text{psinc}_{64}\left(\frac{2\pi}{64}(k - 7.5)\right) e^{-j\left(\frac{2\pi}{64}(k-7.5)\right)\frac{63}{2}}$

$+ \text{psinc}_{64}\left(\frac{2\pi}{64}(k - 56.5)\right) e^{-j\left(\frac{2\pi}{64}(k-56.5)\right)\frac{63}{2}} \Big\}$

$k = 0, \ldots, 63$ 3pt.

[5 Pts] (d) There is truncation of $x[n]$ 1pt

$\Rightarrow$ this example does experience leakage 1pt

In (c), $\omega_0 = \frac{2\pi}{64}\left(\frac{15}{2}\right) = \frac{2\pi}{64}\cdot 7.5$

ω_0 is not multiple of $\frac{2\pi}{64}$ $\Rightarrow$ there is picket fence effects

1.5pt 1.5pt.

4. (25 pts) Consider a random variable X with density function

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases} .$$

a. (3) Carefully sketch *by hand* the density function $f_X(x)$. Be sure to dimension both axes.

b. (8) Find the mean and variance of X.

c. (6) Suppose we generate a new random variable $Y = Q(X)$ by quantizing X according to the following 2-level *uniform* quantizer:

$$Q(x) = \begin{cases} \frac{1}{4}, & 0 \le x < \frac{1}{2} \\ \frac{3}{4}, & \frac{1}{2} \le x < 1 \end{cases} .$$

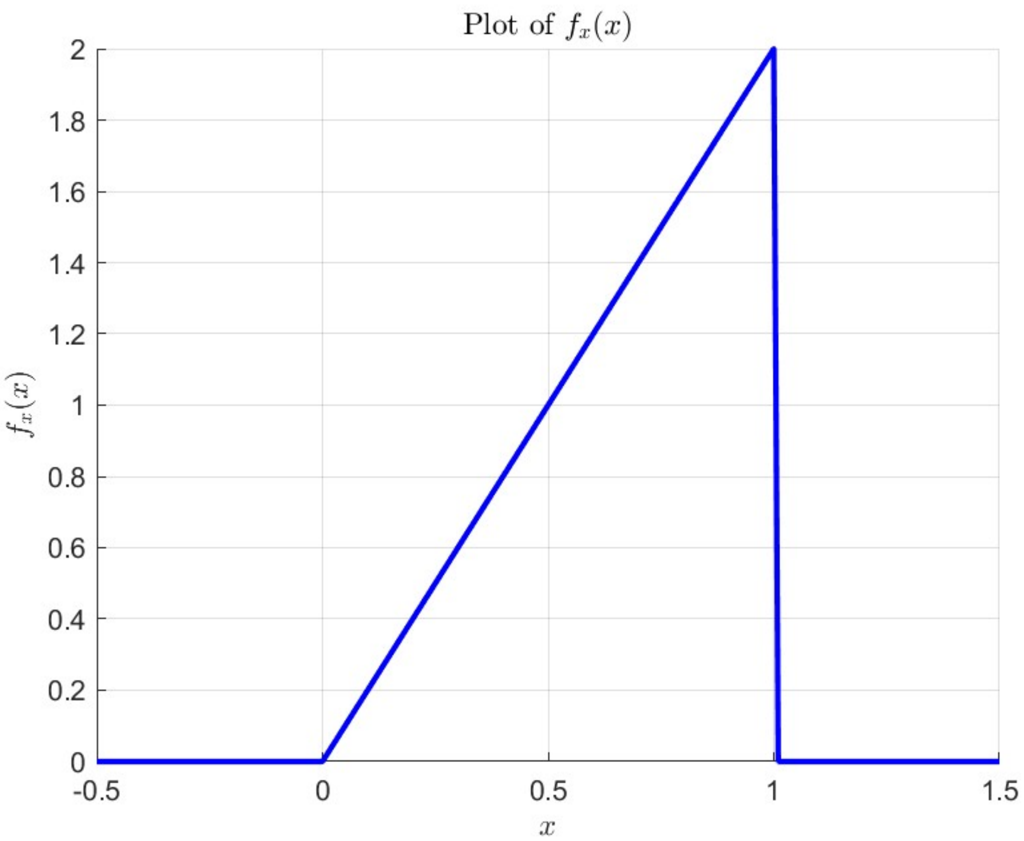
Determine the *approximate* mean-squared quantization error $\varepsilon = E\left\{ \left| Y - X \right|^2 \right\}$ using the expression $\varepsilon_{\text{approx}} = \dfrac{\Delta^2}{12}$

d. (8) Now, provide an expression for the *exact* mean-squared quantization error $\varepsilon_{\text{exact}} = E\left\{ \left| Y - X \right|^2 \right\}$ for the 2-level *uniform* quantizer given above. Your expression should explicitly show the dependence on the quantizer threshold and output levels. However, please do **not** evaluate the integrals. This takes too much algebra.

Problem 4

a.

The plot of $f_x(x)$ is :



Horizontal axis (1pt)

Vertical axis (1pt)

Correct shape. (1pt)

b.

Mean:

$$\begin{aligned}\mathbb{E}\{X\} &= \int_{-\infty}^{\infty} x \cdot f_x(x)\ dx \\ &= \int_0^1 x \cdot 2x\ dx \\ &= \int_0^1 2x^2\ dx \\ &= \frac{2}{3}x^3\Big|_0^1 \\ &= \frac{2}{3} \quad \color{red}{(3pts)}\end{aligned}$$

Variance:

$$\sigma_x^2 = \mathbb{E}\left\{X^2\right\} - (\mathbb{E}\{X\})^2$$

where

$$\begin{aligned}\mathbb{E}\left\{X^2\right\} &= \int_{-\infty}^{\infty} x^2 \cdot f_x(x)\ dx \\ &= \int_0^1 x^2 \cdot 2x\ dx \\ &= \int_0^1 2x^3\ dx \\ &= \frac{1}{2}x^4\Big|_0^1 \\ &= \frac{1}{2} \quad \color{red}{(3pts)}\end{aligned}$$

Therefore:

$$\begin{aligned}\sigma_x^2 &= \mathbb{E}\left\{X^2\right\} - (\mathbb{E}\{X\})^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18} \quad \color{red}{(2pts)}\end{aligned}$$

c.

For the given 2-level quantizer:

$$\begin{aligned}\Delta &= \frac{Q_{max} - Q_{min}}{\text{number of level} - 1} \\ &= \frac{\frac{3}{4} - \frac{1}{4}}{2 - 1} \\ &= \frac{1}{2} \quad \color{red}{(3pts)}\end{aligned}$$

Therefore:

$$\begin{aligned}\epsilon_{approx} &= \frac{\Delta^2}{12} \\ &= \frac{1}{48} \quad \color{red}{(3pts)}\end{aligned}$$

d.

Method (1)

$$
\begin{aligned}
\epsilon_{exact} &= \mathbb{E}\left\{|Y-X|^2\right\} \\
&= \int_{-\infty}^{\infty} (Q(x)-x)^2 \cdot f_x(x)\ dx \quad \color{red}{(4pts)} \\
&= \int_0^{\frac{1}{2}} (\frac{1}{4}-x)^2 \cdot 2x\ dx + \int_{\frac{1}{2}}^{1} (\frac{3}{4}-x)^2 \cdot 2x\ dx \quad \color{red}{(4pts)} \\
&= \int_0^{\frac{1}{2}} (x^2-\frac{1}{2}x+\frac{1}{16}) \cdot 2x\ dx + \int_{\frac{1}{2}}^{1} (x^2-\frac{3}{2}x+\frac{9}{16}) \cdot 2x\ dx \\
&= \int_0^{\frac{1}{2}} (2x^3-x^2+\frac{1}{8}x)\ dx + \int_{\frac{1}{2}}^{1} (2x^3-3x^2+\frac{9}{8}x)\ dx \\
&= \frac{1}{2}x^4\Big|_0^{\frac{1}{2}} - \frac{1}{3}x^3\Big|_0^{\frac{1}{2}} + \frac{1}{16}x^2\Big|_0^{\frac{1}{2}} + \frac{1}{2}x^4\Big|_{\frac{1}{2}}^{1} - x^3|_{\frac{1}{2}}^{1} + \frac{9}{16}x^2\Big|_{\frac{1}{2}}^{1} \\
&= \frac{1}{64} \cdot (-\frac{8}{3}+77-73) \\
&= \frac{1}{64} \cdot (\frac{4}{3}) \\
&= \frac{1}{48}
\end{aligned}
$$

The approximate and the exact quantization errors are the same.

Method (2)

$$\begin{aligned}\epsilon_{exact} &= \mathbb{E}\left\{|Y-X|^2\right\} \\ &= \mathbb{E}\left\{Y^2 - 2XY + X^2\right\} \\ &= \mathbb{E}\left\{Y^2\right\} - 2\mathbb{E}\left\{XY\right\} + \mathbb{E}\left\{X^2\right\} \quad (2pts)\end{aligned}$$

We know $\mathbb{E}\left\{X^2\right\} = \frac{1}{2}$ from problem (b). Now we need to calculate $\mathbb{E}\left\{Y^2\right\}$ and $\mathbb{E}\left\{XY\right\}$.

$$\begin{aligned}\mathbb{E}\left\{Y^2\right\} &= \int_0^{\frac{1}{2}} (\frac{1}{4})^2 \cdot 2x \; dx + \int_{\frac{1}{2}}^1 (\frac{3}{4})^2 \cdot 2x \; dx \quad (3pts) \\ &= \frac{1}{16}x^2\Big|_0^{\frac{1}{2}} + \frac{9}{16}x^2\Big|_{\frac{1}{2}}^1 \\ &= \frac{7}{16}\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left\{XY\right\} &= \int_0^{\frac{1}{2}} (\frac{1}{4}x) \cdot 2x \; dx + \int_{\frac{1}{2}}^1 (\frac{3}{4}x) \cdot 2x \; dx \quad (3pts) \\ &= \frac{1}{6}x^3\Big|_0^{\frac{1}{2}} + \frac{1}{2}x^3\Big|_{\frac{1}{2}}^1 \\ &= \frac{1}{24}\end{aligned}$$

Therefore:

$$\begin{aligned}\epsilon_{exact} &= \mathbb{E}\left\{Y^2\right\} - 2\mathbb{E}\left\{XY\right\} + \mathbb{E}\left\{X^2\right\} \\ &= \frac{7}{16} - 2 \cdot \frac{1}{24} + \frac{1}{2} \\ &= \frac{1}{48}\end{aligned}$$

The approximate and the exact quantization errors are the same.