16 February 2022 **Name:____________________**

**ECE 438** **Exam No. 1** **Spring 2022**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. Smart watches and mobile phones must be put away.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n] - x[n-2].$$

a. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

b. (5) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

c. (5) Find a simple expression for the phase $\underline{/H(\omega)}$ of the frequency response.

d. (5) Carefully sketch $|H(\omega)|$ and $\underline{/H(\omega)}$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

1. Main Approach

$$y[n] = x[n] - x[n-2]$$

Let $x[n] = e^{j\omega n}$

$$y[n] = \left[e^{j\omega n} - e^{j\omega(n-2)}\right]$$

$$= 1\left[e^{j\omega n} - e^{j\omega n} e^{-j\omega 2}\right]$$

$$= e^{j\omega n}\left[1 - e^{-j\omega 2}\right]$$

$$= \left[1 - e^{-j\omega 2}\right] x[n]$$

$$= H(\omega)\, x[n]$$

Simplifying Further using Eulers Formula

$$H(\omega) = 1 - e^{-j\omega 2}$$

$$= e^{-j\omega}\left(e^{j\omega} - e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(j2\sin(\omega)\right)$$

$$= j2e^{-j\omega}\sin(\omega)$$

Rubric. 3 pts: Correct expression for $y[n]$ using $x[n] = e^{j\omega n}$

5 pts : Correct Expression of $H(\omega)$ before Simplifying

2 pts : Correct Expression of $H(\omega)$ after Simplifying

Below are two additional methods to deriving $H(\omega)$

1. $y[n] = x[n] - x[n-2]$

a. There are two methods to compute $H(\omega)$

$$y[n] = x[n] - x[n-2]$$

$$\updownarrow \mathcal{F}$$

$$Y(\omega) = X(\omega) - X(\omega)e^{-j\omega 2}$$

$$Y(\omega) = X(\omega)(1 - e^{-j\omega 2})$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$= 1 - e^{-j\omega 2}$$

Using Euler's Formula, we can simplify further

$$H(\omega) = e^{-j\omega}(e^{j\omega} - e^{-j\omega})$$

$$= e^{-j\omega}(j2\sin(\omega))$$

$$= j2e^{-j\omega}\sin(\omega)$$

Rubric: 3 pts: Correct Expression for $Y(\omega)$

5pts: Correct Expression of $H(\omega)$ before Simplifying

2pts: Correct Expression of $H(\omega)$ after Simplifying

Method 2:

$h[n]$ is defined as the output when $x[n] = \delta[n]$ So

$$h[n] = \delta[n] - \delta[n-2]$$

$$\Big\updownarrow \mathcal{F}$$

$$H(\omega) = 1 - e^{-j\omega 2}$$

Simplifying further using Euler's Formula

$$H(\omega) = e^{-j\omega}\left(e^{j\omega} - e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(j2\sin(\omega)\right)$$

$$= j2e^{-j\omega}\left(\sin(\omega)\right)$$

Rubric: 3pts: Correct Expression of $h[n]$

5pts: Correct Expression of $H(\omega)$ before Simplification

2pts: Correct Expression of $H(\omega)$ after Simplification

b. $$|H(\omega)| = \left|j2e^{-j\omega}\sin(\omega)\right|$$

$$= |2|\,|j|\,\left|e^{-j\omega}\right|\,|\sin(\omega)|$$

$$= 2\,|\sin(\omega)|$$

$\sin(\omega) \geq 0$ for $0 \leq \omega \leq \pi$

$\sin(\omega) \leq 0$ for $-\pi \leq \omega \leq 0$

Rubric: 1 pt: Magnitude of a product equals product of magnitudes

1 pt: correct evaluation of complex exponential

1 pt: Correct evaluation of $j$

1 pt: Determine $\sin(\omega) \geq 0$ for $0 \leq \omega \leq \pi$

1 pt: Correct Final Answer

c. $\angle H(\omega) = \angle\left( j2\, e^{-j\omega} \sin(\omega) \right)$

$\angle H(\omega) = \angle j2 + \angle e^{-j\omega} + \angle \sin(\omega)$

$\angle H(\omega) = \tan^{-1}\left(\frac{2}{0}\right) + -\omega + \tan^{-1}\left(\frac{0}{\sin(\omega)}\right)$

$$= \begin{cases} \frac{\pi}{2} - \omega + 0, & 0 \le \omega \le \pi \\ \frac{\pi}{2} - \omega - \pi, & \pi \le \omega \le 2\pi \quad \text{ie} \quad -\pi \le \omega \le 0 \end{cases}$$

$* \tan^{-1}\left(\frac{0}{\sin(\omega)}\right) = \begin{cases} 0, & 0 \le \omega < \pi \\ \pi \text{ or } -\pi, & -\pi \le \omega \le 0 \end{cases}$

In this case I chose $-\pi$

Rubric: 1pt: Phase of a product equals sum of phases

1pt: Correct evaluation of phase of complex exponential

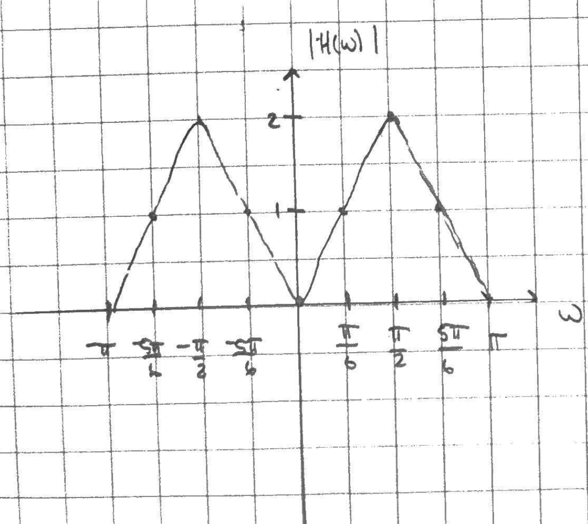
1pt: Correct evaluation of phase of sinusoid

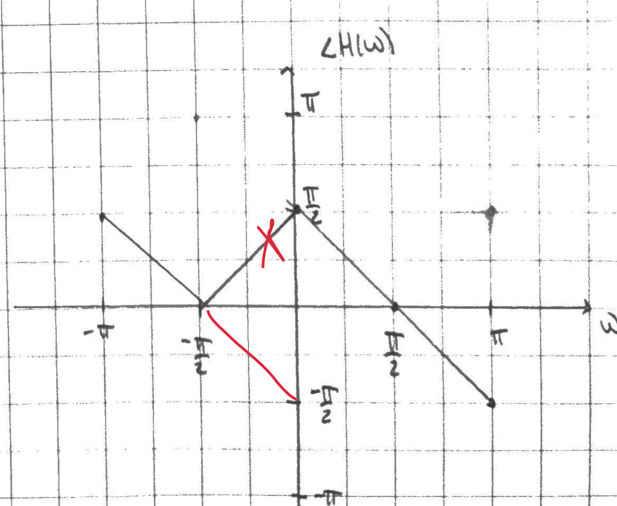
2pt: Correct Final answer

d. Sketching $|H(\omega)|$

Rubric: 1pt: Plot $|H(\omega)|$
1pt: Plot $\angle H(\omega)$
1pt: x-axis label
1pt: y-axis label
1pt: Important Quantities

| | | |
|---|---|---|
| $\omega = 0$ | $\lvert H(0) \rvert = 2\sin(0)$ | $= 0$ |
| $\omega = \pi/6$ | $\lvert H(\pi/6) \rvert = 2\sin\left(\frac{\pi}{6}\right)$ | $= 1$ |
| $\omega = \pi/2$ | $\lvert H(\pi/2) \rvert = 2\sin\left(\frac{\pi}{2}\right)$ | $= 2$ |
| $\omega = 5\pi/6$ | $\lvert H(5\pi/6) \rvert = 2\sin\left(\frac{5\pi}{6}\right)$ | $= 1$ |
| $\omega = \pi$ | $\lvert H(\pi) \rvert = 2\sin(\pi)$ | $= 0$ |



2. (25 pts.) Consider a linear, time-invariant system with unit sample (impulse) response

$$h[n]=\begin{cases} 1, & 0\le n\le 4 \\ 0, & \text{else} \end{cases} .$$

a. (20) Find the response of this system $y[n]$ to the input

$$x[n]=\left(\frac{1}{3}\right)^n u[n],$$

by evaluating the convolution $y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ .

Your solution for $y[n]$ should be an analytical expression or expression(s) for the signal. It should not contain any summation signs $\sum$ .

b. (5) Carefully sketch the function $y[n]$ .

2. (continued - 2)

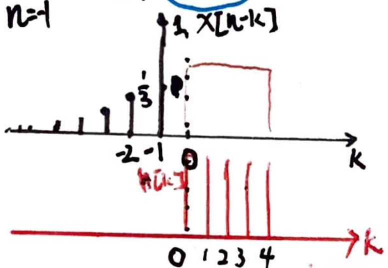
Solution: a) $x[n] = \left(\frac{1}{3}\right)^n u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=0}^{4} x[n-k] = \sum_{k=0}^{4} x[-(k-n)]$$

Case 1: $n<0$

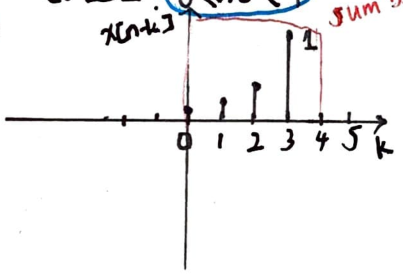
eg: $n=-1$




$$y[n] = 0$$

Case 2: $0 \leq n < 4$

eg: $n=3$




$$y[n] = \sum_{k=0}^{4} \left(\frac{1}{3}\right)^{n-k} u[n-k]$$

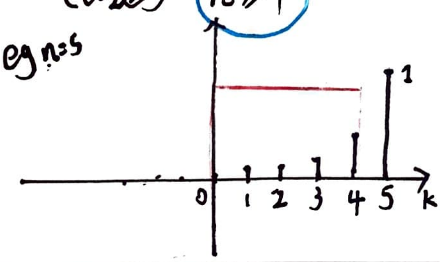
$$= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^{n-k} = \sum_{k=0}^{n} 3^{k-n}$$

$$= 3^{-n} \frac{1-3^{n+1}}{1-3} = \frac{3^{n+1}-1}{2} \cdot 3^{-n}$$

$$= \frac{3}{2} - \frac{3^{-n}}{2}$$

case 3 $n \geq 4$

eg $n=5$



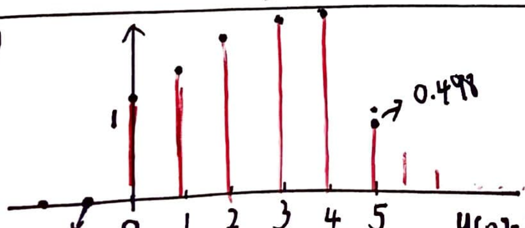

$$y[n] = \sum_{k=0}^{4} \left(\frac{1}{3}\right)^{n-k} u[n-k] = \sum_{k=0}^{4} \left(\frac{1}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{4} 3^{k-n}$$

$$= 3^{-n} \frac{1-3^5}{1-3} = \frac{3^5-1}{2} \cdot 3^{-n} = \frac{3^{5-n}}{2} - \frac{3^{-n}}{2}$$

$$= 121 \left(\frac{1}{3}\right)^n$$

b)




$y[0]=1$, $y[1] = \frac{4}{3} = 1.333$, $y[2] = 13/9 = 1.444$, $y[3] = 40/27 = 1.481$, $y[4] = 1.444$

3. (25) Consider the real-valued continuous-time signal $x(t)$ defined by

$$x(t)=\begin{cases} \frac{1}{2}\left(1+\cos(2\pi t/10)\right), & |t|<5 \\ 0, & \text{else} \end{cases}$$

a. (5) Carefully sketch $x(t)$ being sure to dimension both axes.

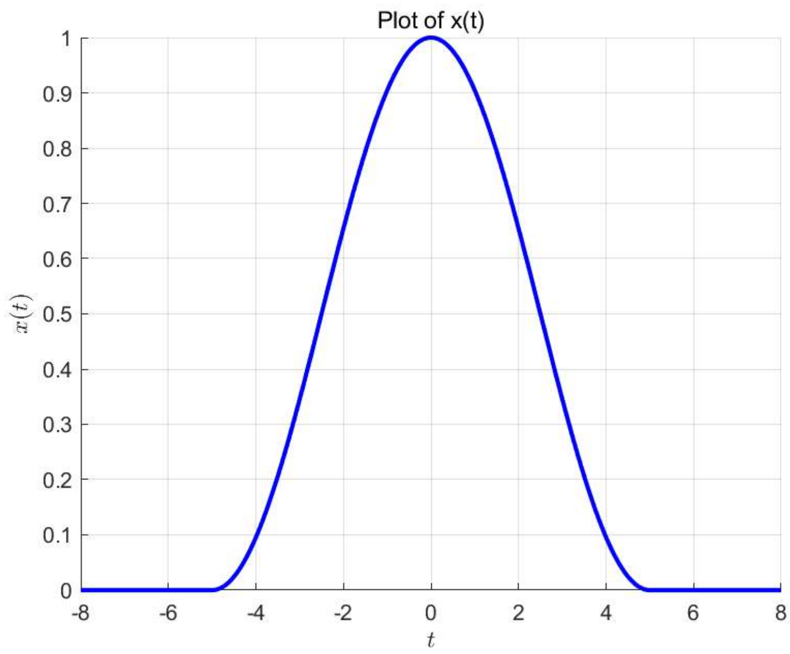
b. (15) Find a simple expression for the CTFT $X(f)$ of $x(t)$. Your answer should not include any operators, such as convolution, rep, or comb.

c. (5) Carefully sketch $X(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes

# Problem 3

## a.

The plot of $x(t)$ is shown here:




Horizontal axis (1pt)

Vertical axis (1pt)

Plot of $x(t)$ (3pts)

## b.

$$x(t) = \text{rect}(\frac{t}{10}) \cdot \frac{1}{2}\left[1 + cos(\frac{2\pi t}{10})\right]$$

The CTFT of $x(t)$ is:

$$\begin{aligned}
X(f) &= 10\text{sinc}(10f) * \frac{1}{2}\left\{\delta(f) + \frac{1}{2}\left[\delta(f - \frac{1}{10}) + \delta(f + \frac{1}{10})\right]\right\} \\
&= 5\text{sinc}(10f) * \left\{\delta(f) + \frac{1}{2}\left[\delta(f - \frac{1}{10}) + \delta(f + \frac{1}{10})\right]\right\} \\
&= 5\text{sinc}(10f) + \frac{5}{2}\left[\text{sinc}(10(f - \frac{1}{10})) + \text{sinc}(10(f + \frac{1}{10}))\right]
\end{aligned}$$

Expression of $x(t)$ (3pts)

CTFT of rect($\frac{t}{10}$) (2.5pts)

CTFT of 1 (2.5pts)

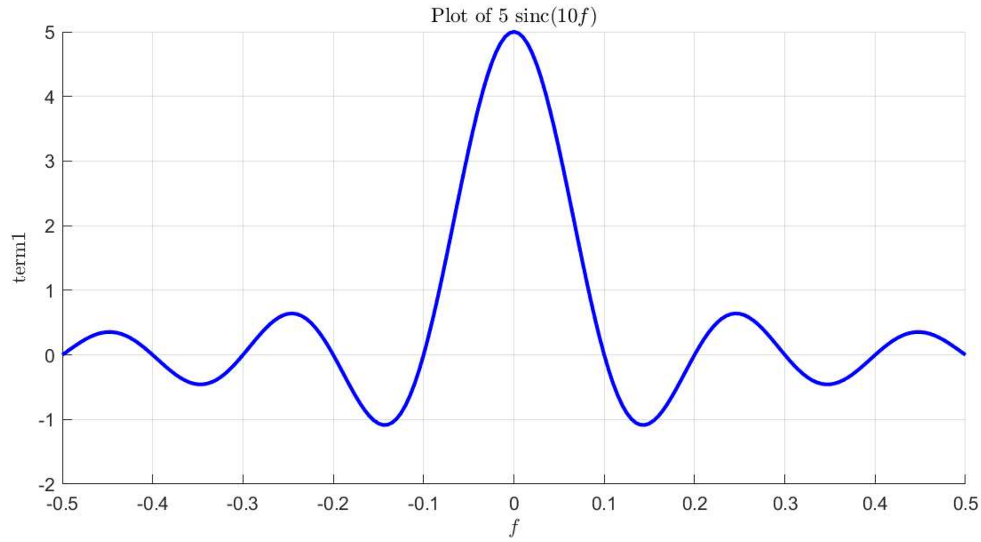
CTFT of $cos(\frac{2\pi t}{10})$ (2.5pts)

Multiplication in time domain is convolution in frequency domain (2.5pts)

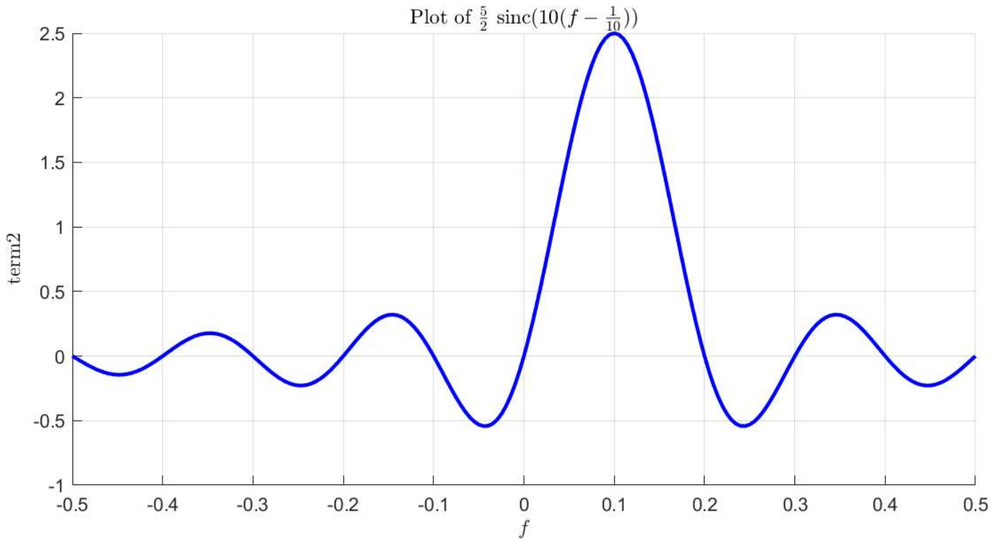
A correct final answer (2pts)

**c.**

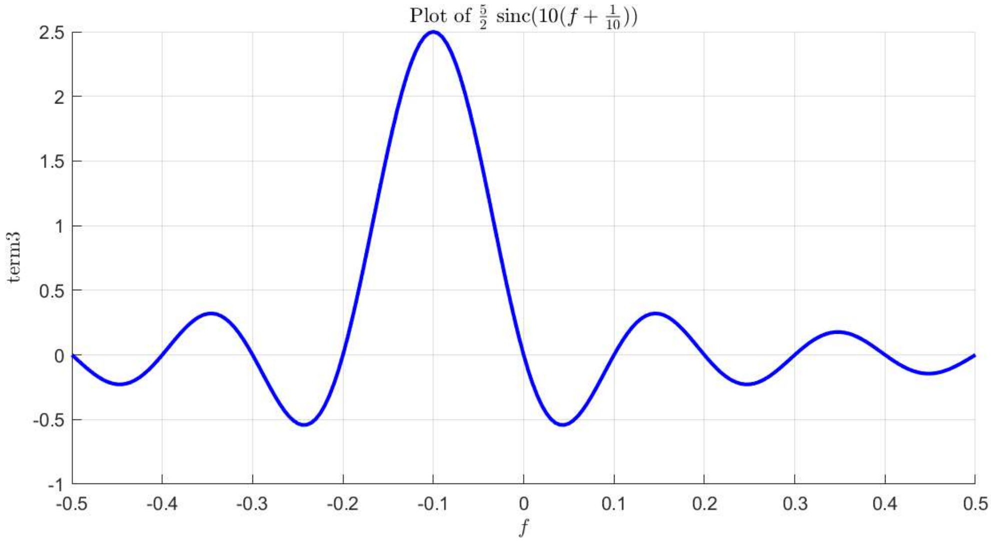
The plot of $5\text{sinc}(10f)$ is :



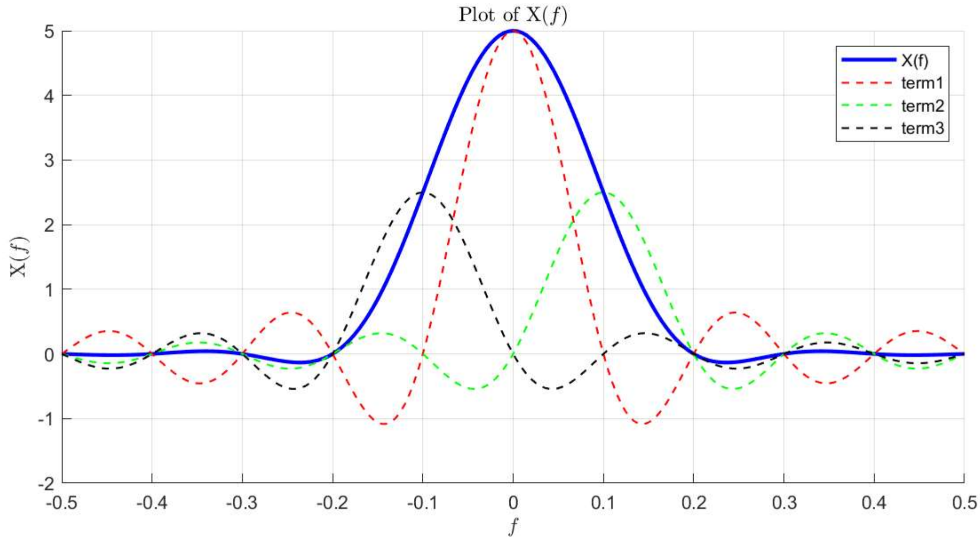

The plot of $\frac{5}{2}\text{sinc}(10(f-\frac{1}{10}))$ is :



The plot of $\frac{5}{2}\text{sinc}(10(f+\frac{1}{10}))$ is :




So the overall plot of $X(f)$ is:

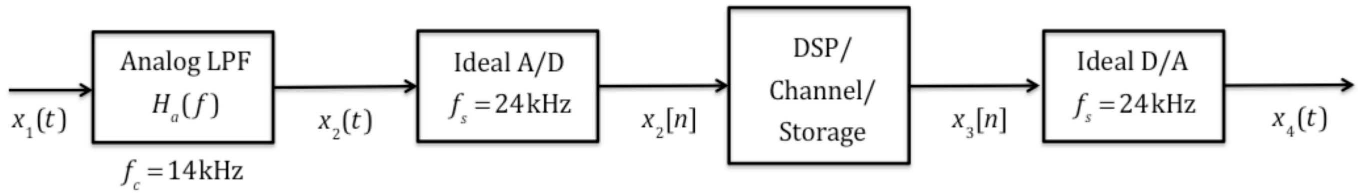



Horizontal axis (1pt)

Vertical axis (1pt)

A correct plot of $X(f)$ or three correct sub-plots. (3pts)

4. (25 pts) (a) (15 pts.) Consider the system shown below:



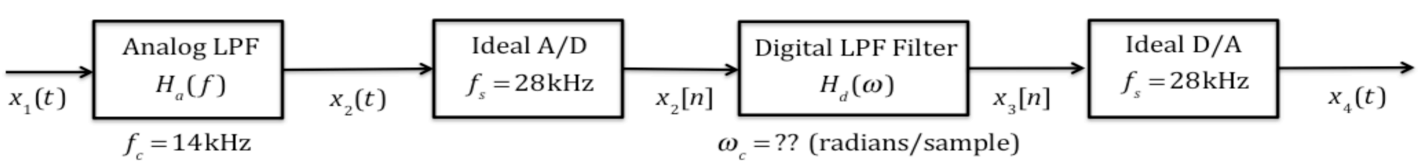

Today, with inflation reaching levels not seen in many years, vendors are deploying various tricks to reduce their costs, while hoping that consumers will not notice. An example is reducing portion sizes in food items purchased at restaurants or in grocery stores.

Your task is to cost-reduce the system shown above to capture audio signals $x_1(t)$, convert them to digital format $x_2[n]$, process, transmit, or store them, and then convert the output $x_3[n]$ to analog format $x_4(t)$ for listening by the user.

To reduce the overall cost of the system, you decide to reduce the sampling frequency from 28 kHz to 24 kHz, thus achieving a savings of 14.29%. But you decide not to change the analog low-pass prefilter, since it is hard-wired. So its frequency response $H_a(f)$ continues to have cutoff $f_c$ = 14 kHz. But will it work?

Find the output $x_4(t)$ from this system assuming that the input is $x_1(t) = \cos\left(2\pi(13000)t\right)$. Discuss your results. Your answer should be simplified as much as possible. You may ignore the box labeled DSP/Channel/Storage; so $x_3[n] \equiv x_2[n]$.

(b) (10 pts.) Professor Allebach has a hearing disability. He basically cannot hear sounds at frequencies above 2 kHz. You decide to design a digital system that will allow you to hear sounds as he hears them. The system is shown below:




Your task is to choose the cutoff frequency $\omega_c$ for the digital LPF filter so that the output $x_4(t)$ resembles how $x_1(t)$ would sound to Professor Allebach. Discuss your results. Your answer should be simplified as much as possible.

4.

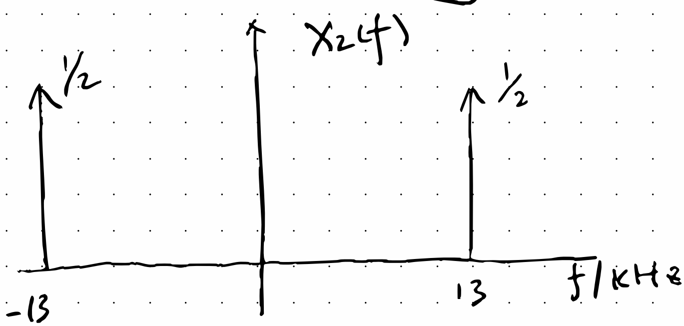
(a) It won't work. 2 pts

The sampling frequency (24 kHz) is less than twice than the maximum frequency (14 kHz × 2 = 28 kHz).

$x_1(t) = \cos(2\pi(13000)t)$

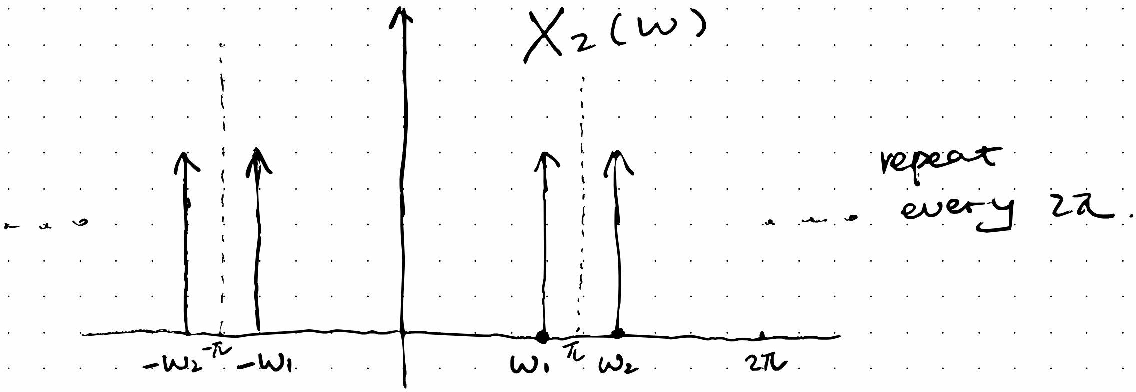
$x_2(t) = \cos(2\pi(13000)t)$ 1 pts

Analog LPF has no effect because the frequency of $x_1(t)$ is 13 kHz and the cut-off frequency of the analog LPF is 14 kHz.




$X_2(f) = \frac{1}{2}\{\delta(f-13000) + \delta(f+13000)\}$

1 pts




here: $w_1 = 2\pi - \frac{13}{24} \cdot 2\pi = \frac{11}{12}\pi$

$w_2 = \frac{13}{24} \cdot 2\pi = \frac{13}{12}\pi$

$$X_2(\omega) = f_s \, \text{rep}_{f_s} \left| X_2(f) \right|_{f = \frac{\omega}{2\pi} f_s}$$ 1 pts

$$= f_s \sum_{k=-\infty}^{\infty} \frac{1}{2} \left( \delta\left(\frac{\omega}{2\pi} f_s - 13000 - 13000k\right) + \delta\left(\frac{\omega}{2\pi} f_s + 13000 - 13000k\right) \right)$$

$\delta(at - t_0) = \frac{1}{|a|} \delta\left(t - \frac{t_0}{a}\right)$

$$= \sum_{k=-\infty}^{\infty} \pi \delta\left(\omega - 2\pi \frac{13000}{24000} - 2\pi k\right) + \pi \delta\left(\omega + 2\pi \frac{13000}{24000} - 2\pi k\right)$$

$$= \sum_{k=-\infty}^{\infty} \pi \delta\left(\omega - \frac{13}{12}\pi - 2\pi k\right) + \pi \delta\left(\omega + \frac{13}{12}\pi - 2\pi k\right)$$ 1 pts

aliasing occured

$$= \sum_{k=-\infty}^{\infty} \pi \delta\left(\omega + \frac{11}{12}\pi - 2\pi k\right) + \pi \delta\left(\omega - \frac{11}{12}\pi - 2\pi k\right)$$

(plot above)

2 pts

$$X_4(f) = \frac{1}{f_s} X_2(\omega)\Big|_{\omega=\frac{2\pi f}{f_s}} \times \mathrm{rect}\left(\frac{f}{f_s}\right)$$

1 pts

$$= \left( \frac{1}{f_s} \sum_{k=-\infty}^{\infty} \pi\delta\left(\frac{2\pi f}{f_s} + \frac{11}{12}\pi - 2\pi k\right) + \pi\delta\left(\frac{2\pi f}{f_s} - \frac{11}{12}\pi - 2\pi k\right) \right)$$

$$\times \mathrm{rect}\left(\frac{f}{f_s}\right)$$

$$= \left( \frac{1}{2} \sum_{k=-\infty}^{\infty} \delta\left(f + \frac{11}{24} f_s - k f_s\right) + \delta\left(f - \frac{11}{24} f_s - k f_s\right) \right)$$

$$\times \mathrm{rect}\left(\frac{f}{f_s}\right)$$

$$= \frac{1}{2}\left\{ \delta(f + 11000) + \delta(f - 11000) \right\}$$

$$x_4(t) = \cos(2\pi(11000)t)$$

1 pts

5 pts : correct answer

(b)

The cut-off frequency is 2kHz

$$\omega_c = \frac{f_c}{f_s} \times 2\pi$$

5 pts

$$= \frac{2}{28} \times 2\pi$$

$$= \frac{1}{7}\pi \text{ radians/sample}$$

4 pts

1 pt

correct answer