6 May 2020 **Name:** Solution **Signature:** ____________

ECE 438 **Final Exam** **Spring 2020**

- You have 48 hours to work the following five problems, which are worth a total of 125 points. You should e-mail a scan or images of your solution to ece438@ecn.purdue.edu by 1:00p EDT on Friday 8 May.
- Be sure to show all your work to obtain full credit.
- The exam is open book and open notes.
- Please do **NOT** discuss the problems with anyone else.

1. (20 pts.) Consider a discrete-time, linear, time-invariant system with unit sample response $h[n]$.

   a. (5) Find a general expression for the frequency response $H(\omega)$ of this system in terms of $h[n]$.

   b. (5) From your answer to part (a), show that if the unit sample response $h[n]$ is real-valued, then $H^*(\omega) = H(-\omega)$, where the asterisk denotes the complex conjugate.

   c. (5) From your answer to part (b), show that if the unit sample response $h[n]$ is real-valued, then $|H(\omega)| = |H(-\omega)|$ and $\angle H(\omega) = -\angle H(-\omega)$, i.e. the magnitude is an even function of ω and the phase is an odd function of ω.

   d. (5) From your answer to part (c) and the definition of frequency response $H(\omega)$ as the response of the system to the complex exponential signal $x[n] = e^{j\omega n}$, show that the response $y_0[n]$ to the signal $x_0[n] = \cos(\omega_0 n)$ can be expressed as

$$y_0[n] = |H(\omega_0)| \cos\left(\omega_0 n + \angle H(\omega_0)\right).$$

Problem 1 (20)

Consider a discrete-time, linear, time-invariant system with unit sample response $h[n]$.

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b. (5) From your answer to part (a), show that if the unit sample response $h[n]$ is real-valued, then $H^*(\omega) = H(-\omega)$, where the asterisk denotes the complex conjugate.

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d. (5) From your answer to part (c) and the definition of frequency response $H(\omega)$ as the response of the system to the complex exponential signal $x[n] = e^{j\omega n}$, show that the response $y_0[n]$ to the signal $x_0[n] = cos(\omega_0 n)$ can be expressed as

$$y_0[n] = |H(\omega_0)| cos\left(\omega_0 n + \underline{/H(\omega_0)}\right).$$

a.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

5 pts

Method 1 for (b), (c), and (d):

b.

$$\begin{aligned} H^*(\omega) &= \left[\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}\right]^* \\ &= \sum_{n=-\infty}^{\infty} h^*[n] e^{j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n] e^{-j(-\omega)n} \\ &= H(-\omega) \end{aligned}$$

3 pts: complex conjugate of H(w) from part (a)
2 pts: H(-w)

c. We can write

$$\begin{aligned} H^*(\omega) &= \left[|H(\omega)| e^{j\underline{/H(\omega)}}\right]^* \\ &= |H(\omega)| e^{-j\underline{/H(\omega)}} \\ &= |H(\omega)| e^{j\left(-\underline{/H(\omega)}\right)} \end{aligned} \quad (1)$$

2 pt: complex conjugate of H(w) in polar coordinate

and

$$H(-\omega) = |H(-\omega)| e^{j\underline{/H(-\omega)}} \quad (2)$$

1 pt: H(-w) in polar coordinate

Since $H^*(\omega) = H(-\omega)$ from part (b), we can equate magnitude and phase components of Equas (1) and (2). We have:

2 pts: equating magnitude and phase components

$$|H(\omega)| = |H(-\omega)|$$

$$\underline{/H(\omega)} = -\underline{/H(-\omega)}$$

d. By Euler's identity, we have

1 pt: use Euler's identity to express x0[n] as two complex exponential signals

$$cos(\omega_0 n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) \tag{3}$$

Since the system is linear and by definition of the frequency response of the system as the response to a complex exponential input, we have

1 pt: find the output of the first complex exponential part exp(jw0n)

$$\begin{aligned} e^{j\omega_0 n} &\xrightarrow{\text{System}} H(\omega_0)e^{j\omega_0 n} \\ &\xrightarrow{\text{System}} |H(\omega_0)|e^{j\underline{/H(\omega_0)}}e^{j\omega_0 n} \\ &\xrightarrow{\text{System}} |H(\omega_0)|e^{j\left[\omega_0 n + \underline{/H(\omega_0)}\right]} \end{aligned} \tag{4}$$

and

2 pts: find the output of the second complex exponential part exp(-jw0n), using answers to part(c)

$$\begin{aligned} e^{-j\omega_0 n} &\xrightarrow{\text{System}} H(-\omega_0)e^{-j\omega_0 n} \\ &\xrightarrow{\text{System}} |H(-\omega_0)|e^{j\underline{/H(-\omega_0)}}e^{-j\omega_0 n} \\ &\xrightarrow{\text{System}} |H(\omega_0)|e^{-j\underline{/H(\omega_0)}}e^{-j\omega_0 n} \quad \text{(from part(c))} \\ &\xrightarrow{\text{System}} |H(\omega_0)|e^{-j\left[\omega_0 n + \underline{/H(\omega_0)}\right]} \end{aligned} \tag{5}$$

Summing Eqs (4) and (5), and multiplying by $\frac{1}{2}$, according to (3), we get

$$\begin{aligned} cos(\omega_0 n) &\xrightarrow{\text{System}} \left\{\frac{1}{2}|H(\omega_0)|e^{j\left[\omega_0 n + \underline{/H(\omega_0)}\right]} + |H(\omega_0)|e^{-j\left[\omega_0 n + \underline{/H(\omega_0)}\right]}\right\} \\ &\xrightarrow{\text{System}} \frac{1}{2}|H(\omega_0)|\left\{e^{j\left[\omega_0 n + \underline{/H(\omega_0)}\right]} + e^{-j\left[\omega_0 n + \underline{/H(\omega_0)}\right]}\right\} \\ &\xrightarrow{\text{System}} |H(\omega_0)|cos\left(\omega_0 n + \underline{/H(\omega_0)}\right) \end{aligned}$$

1 pt: using Euler's identity to get the final form.

Method 2 for (b), (c), and (d):

b. Since $h[n]$ is real-valued,

$$Re\{H(\omega)\} = \sum_{n=-\infty}^{\infty} h[n]cos\omega n, \quad Im\{H(\omega)\} = -\sum_{n=-\infty}^{\infty} h[n]sin\omega n$$

$$H^*(\omega) = \sum_{n=-\infty}^{\infty} h[n]cos\omega n + j\sum_{n=-\infty}^{\infty} h[n]sin\omega n$$
$$= \sum_{n=-\infty}^{\infty} h[n]e^{j\omega n}$$

3 pts: complex conjugate of H(w)
2 pts: H(-w)

$$H(-\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(-\omega)n} = \sum_{n=-\infty}^{\infty} h[n]e^{j\omega n}$$

Therefore, $H^*(\omega) = H(-\omega)$

c.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]cos\omega n - j\sum_{n=-\infty}^{\infty} h[n]sin\omega n$$

$$H(-\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} h[n]cos\omega n + j\sum_{n=-\infty}^{\infty} h[n]sin\omega n$$

First, check for $|H(\omega)|$ and $|H(-\omega)|$

$$|H(\omega)| = \sqrt{\left(\sum_{n=-\infty}^{\infty} h[n]cos\omega n\right)^2 + \left(-\sum_{n=-\infty}^{\infty} h[n]sin\omega n\right)^2}$$

2 pts: magnitude calculations in Cartesian coordinate

$$|H(-\omega)| = \sqrt{\left(\sum_{n=-\infty}^{\infty} h[n]cos\omega n\right)^2 + \left(\sum_{n=-\infty}^{\infty} h[n]sin\omega n\right)^2}$$

Therefore, $|H(\omega)| = |H(-\omega)|$.

Now, check for $\angle H(\omega)$ and $\angle H(-\omega)$:

$$\angle H(\omega) = arctan\left(-\frac{\sum_{n=-\infty}^{\infty} h[n]sin\omega n}{\sum_{n=-\infty}^{\infty} h[n]cos\omega n}\right)$$

2 pts: angle calculations in Cartesian coordinate.

$$\angle H(-\omega) = arctan\left(\frac{\sum_{n=-\infty}^{\infty} h[n]sin\omega n}{\sum_{n=-\infty}^{\infty} h[n]cos\omega n}\right)$$

1 pt: verify odd function

Since $arctan(x)$ is an odd function of x, $\underline{/H(\omega)} = -\underline{/H(-\omega)}$

d.

$$\begin{aligned}
y_0[n] &= h[n] * x_0[n] \\
&= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \sum_{m=-\infty}^{\infty} h[m]cos(\omega_0(n-m)) \qquad (cos(\alpha-\beta) = cos\alpha cos\beta + sin\alpha sin\beta) \\
&= \sum_{m=-\infty}^{\infty} h[m]\left(cos(\omega_0 n)cos(\omega_0 m) + sin(\omega_0 n)sin(\omega_0 m)\right) \\
&= cos(\omega_0 n) \sum_{m=-\infty}^{\infty} h[m]cos(\omega_0 m) + sin(\omega_0 n) \sum_{m=-\infty}^{\infty} h[m]sin(\omega_0 m)
\end{aligned}$$

recall that $Re\{H(\omega)\} = \sum_{n=-\infty}^{\infty} h[n]cos\omega n, \quad Im\{H(\omega)\} = -\sum_{n=-\infty}^{\infty} h[n]sin\omega n$

$$y_0[n] = cos(\omega_0 n)Re\{H(\omega_0)\} - sin(\omega_0 n)Im\{H(\omega_0)\}$$

2 pts: identify real and imaginary parts of H(w) in the convolution result

Since

$$Re\{H(\omega)\} = |H(\omega)|cos\underline{/H(\omega)}$$

$$Im\{H(\omega)\} = |H(\omega)|sin\underline{/H(\omega)}$$

2 pts: express real and imaginary parts in magnitude and angle form

we can write

$$\begin{aligned}
y_0[n] &= |H(\omega_0)|cos\underline{/H(\omega_0)}cos(\omega_0 n) - |H(\omega_0)|sin\underline{/H(\omega_0)}sin(\omega_0 n) \\
&= |H(\omega_0)|\left(cos\underline{/H(\omega_0)}cos(\omega_0 n) - sin\underline{/H(\omega_0)}sin(\omega_0 n)\right) \\
&= |H(\omega_0)|cos(\omega_0 n + \underline{/H(\omega_0)})
\end{aligned}$$

(since $cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$)

1 pt: using trigonometric identity to get the final form.

2. (30 pts.) Consider the discrete-time, linear, time-invariant system described by the following difference equation

$$y[n] = x[n] + x[n-1] + \frac{1}{2}x[n-2] + y[n-1] - \frac{1}{2}y[n-2]$$

a. (2) Is this system causal? Explain why or why not.

b. (4) Find the transfer function $H(z)$ for this system.

c. (5) Based on your answer to part (b), find the poles and zeros for this system, and plot them in the complex z plane.

d. (2) Is this system bounded-input-bounded-output (BIBO) stable? Explain why or why not.

e. (10) Use the graphical approach introduced in class to find the magnitude of the frequency response $|H(\omega)|$ at the frequencies $\omega = 0, \pi/4, \pi/2, 3\pi/4,$ and π radians/sample. Be sure to show the details of your computation.

f. (3) Based on your answer to part (e), sketch $|H(\omega)|$ for all frequencies $0 \le \omega \le 2\pi$.

g. (4) Use the graphical approach introduced in class to find the phase of the frequency response $\underline{/H(\omega)}$ at the frequencies $\omega = 0$ and $\pi/2$ radians/sample. Be sure to show the details of your computation.

2)

(a) 2 pts: Correct answer

Yes, it only depends on current and past inputs and past outputs. (No terms of the form $x[n+c], c > 0$)

(b)

Recall Z-transform property: $x[n-c] \overset{Z}{\Leftrightarrow} z^{-c}X(z)$

$$Y(z) = X(z) + z^{-1}X(z) + \frac{1}{2}z^{-2}X(z) + z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z)$$

1 pt: Correct Z-transform of y[n]

$$\Rightarrow Y(z) - z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) = X(z) + z^{-1}X(z) + \frac{1}{2}z^{-2}X(z)$$

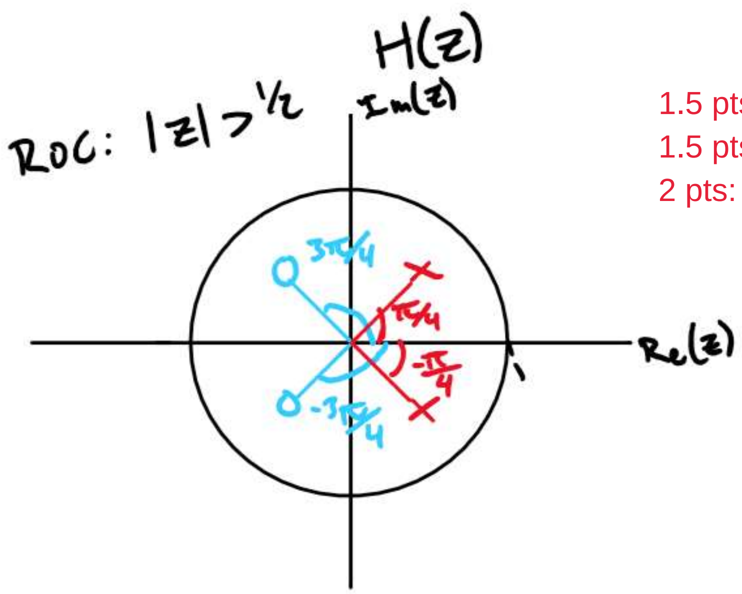
$$\Rightarrow \left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)Y(z) = \left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)}{\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{\left(z - \left(-\frac{1}{2} + \frac{j}{2}\right)\right)\left(z - \left(-\frac{1}{2} - \frac{j}{2}\right)\right)}{\left(z - \left(\frac{1}{2} + \frac{j}{2}\right)\right)\left(z - \left(\frac{1}{2} - \frac{j}{2}\right)\right)}$$

1.5 pts: Correct numerator

1.5 pts: Correct denominator

(c)



1.5 pts: Correctly identify poles

1.5 pts: Correctly identify zeros

2 pts: Correct plot

(d) 2 pts Correct answer

Yes, to be BIBO stable, you must have the unit circle in the ROC of the transfer function, which is true for this system.

(e)

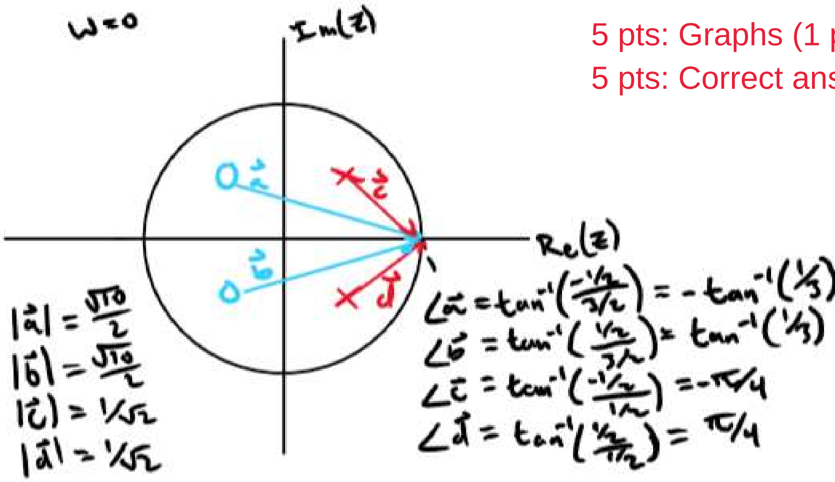
$$|H(\omega)| = \frac{\left|z-\left(-\frac{1}{2}+\frac{j}{2}\right)\right|\left|z-\left(-\frac{1}{2}-\frac{j}{2}\right)\right|}{\left|z-\left(\frac{1}{2}+\frac{j}{2}\right)\right|\left|z-\left(\frac{1}{2}-\frac{j}{2}\right)\right|} = \frac{|a||b|}{|c||d|}$$

Where $a, b, c,$ and d are equal to the expression in their respective position in the equation. The angles are calculated using $\angle a = \tan^{-1}\left(\frac{\text{Im}(a)}{\text{Re}(a)}\right)$.

5 pts: Graphs (1 pt for each w)

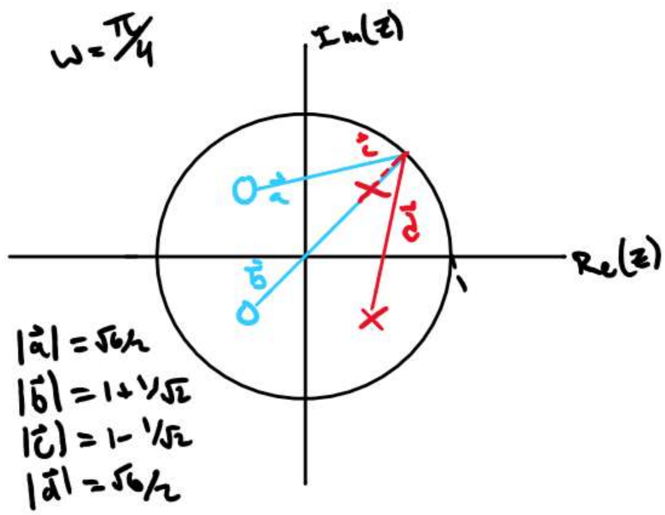
5 pts: Correct answer (1 pt for each w)

$$|\vec{a}| = \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{1}{4}} = \frac{\sqrt{10}}{2}$$

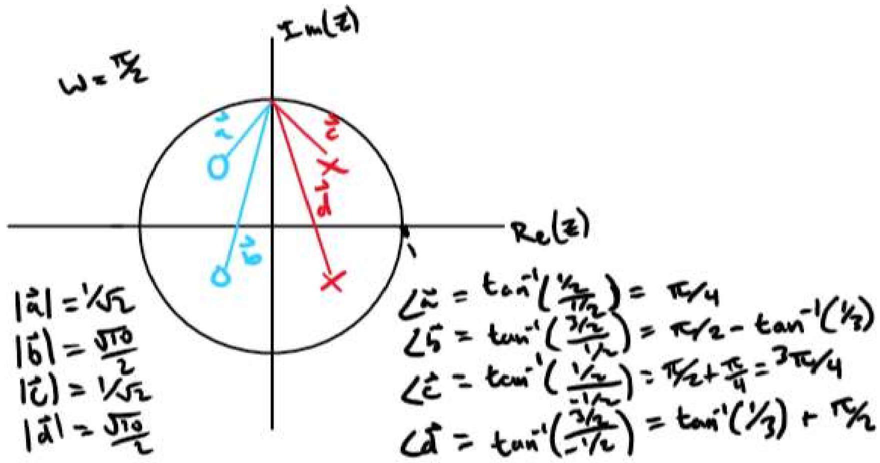


$$|\vec{c}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$$

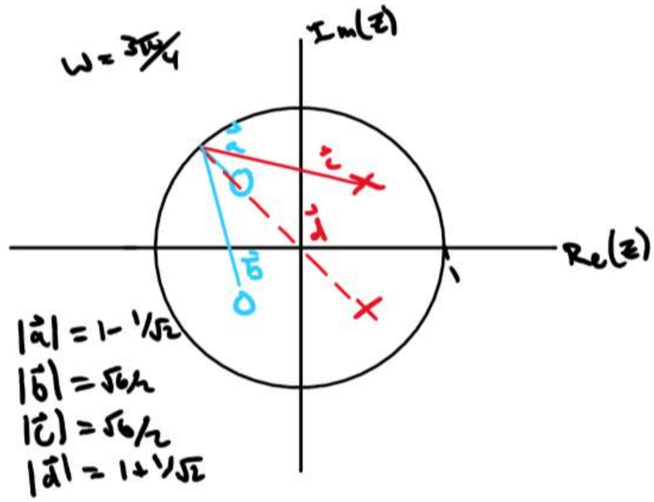
So, $|H(0)| = \frac{5/2}{1/2} = 5$



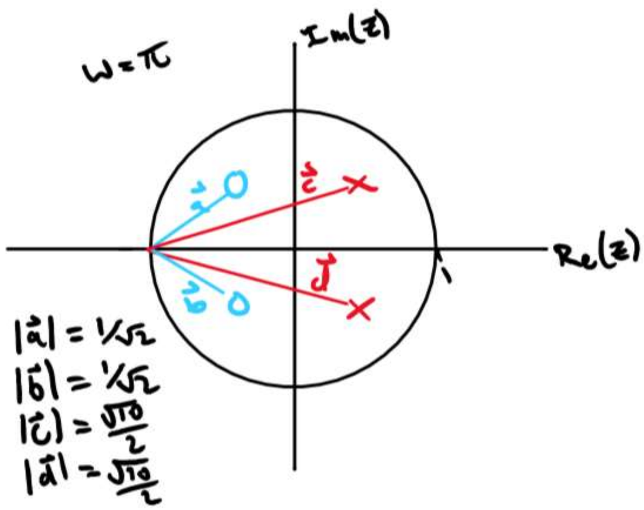
So, $|H(\pi/4)| = \frac{(1+1/\sqrt{2})}{(1-1/\sqrt{2})} = 2\sqrt{2}+3 \approx 5.8$



So, $|H(\pi/2)| = 1$

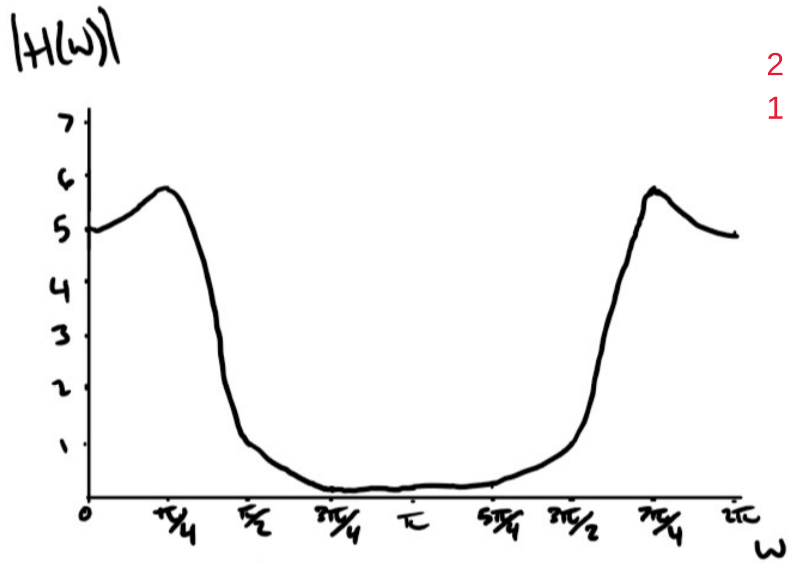


So, $|H(3\pi/4)| = \frac{(1-1/\sqrt{2})}{(1+1/\sqrt{2})} = 3 - 2\sqrt{2} \approx 0.2$



So, $|H(\pi)| = \frac{1/2}{5/2} = \frac{1}{5} = 0.2$

(f)



2 pts: Correct shape
1 pt: Axes labeled

(g)

Using the angles of $a, b, c,$ and d found in part (e) we can calculate the phase for $\omega = 0$ and $\omega = \pi/2$ with the following equation:

/H(ω) = /a + /b - /c - /d

So, /H(0) $= -\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3} - \frac{-\pi}{4} - \frac{\pi}{4} = 0$

and /H(π/2) $= \frac{\pi}{4} + \left(\frac{\pi}{2} - \tan^{-1}\frac{1}{3}\right) - \frac{3\pi}{4} - \left(\frac{\pi}{2} + \tan^{-1}\frac{1}{3}\right) = -\frac{\pi}{2} - 2\tan^{-1}\frac{1}{3}$ = 0.927 radians = 53.13 degrees

1 pt: Graphical approach used
3 pts: Correct answers (1.5 pts each)

3. (25 pts.) Let $X[n]$ be discrete-time random process defined as follows:

$$X[n] = 4\cos\left(\frac{\pi}{4}n + \theta\right), \quad -\infty < n < \infty,$$

where θ is a random variable uniformly distributed on the interval $[-\pi, \pi)$.

a. (5) Sketch three different sample functions from this random process, showing sufficient detail to convince the reader that you know what these sample functions look like.

b. (5) Find the mean of the random process $X[n]$.

c. (12) Find the autocorrelation $r_{XX}[m,n]$ of the random process $X[n]$.

d. (3) Is $X[n]$ a wide-sense stationary random process? Explain why or why not.

Problem 3

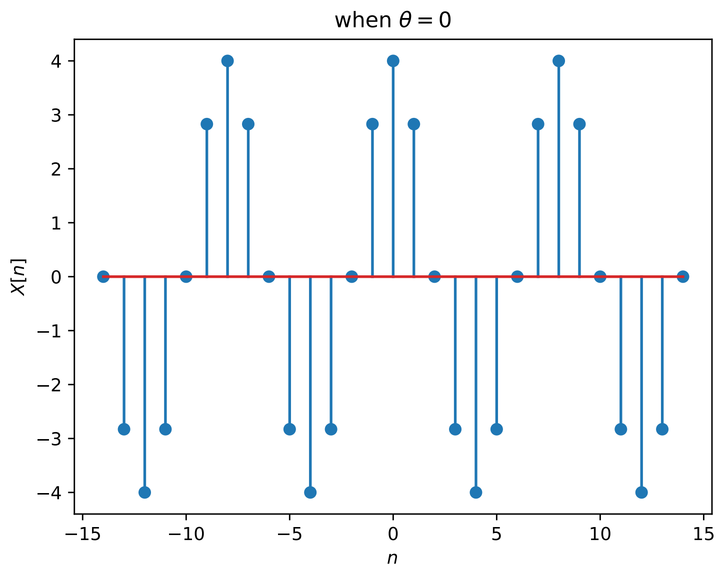
(25 points) Let $X[n]$ be discrete-time random process defined as follows:

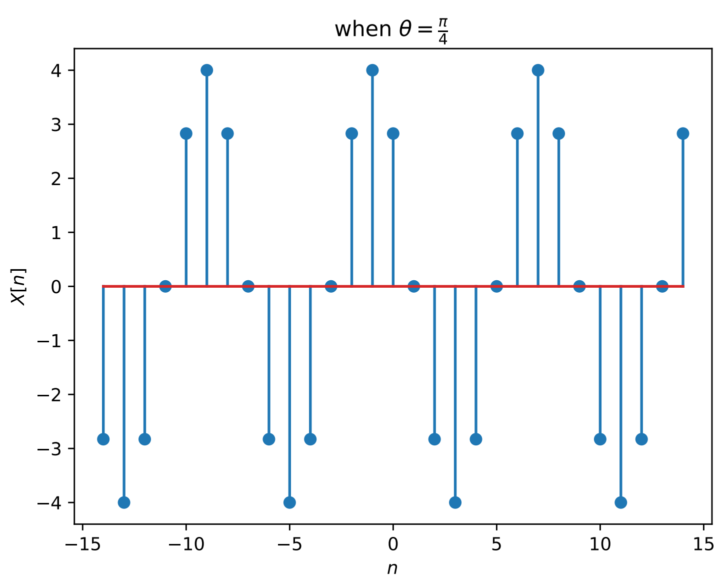
$$X[n] = 4\cos\left(\frac{\pi}{4}n + \theta\right), \quad -\infty < n < \infty$$

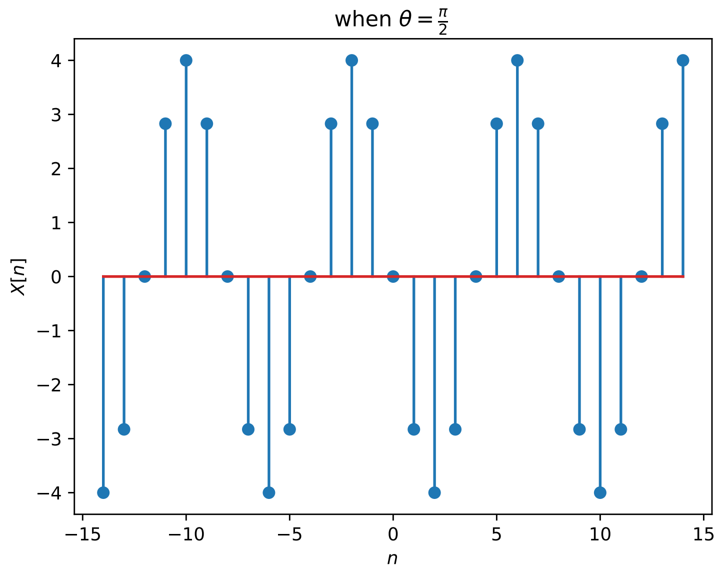
where θ is a random variable uniformly distributed on the interval $[-\pi, \pi)$.

(a) **(5 points)** Sketch three different sample functions from this random process, showing sufficient detail to convince the reader that you know what these sample functions look like.

3 pts: 3 correct plots
2 pts: correct labels







(b) **(5 points)** Find the mean of the random process $X[n]$.

Note that θ is a uniformly distributed random variable on the interval $[-\pi, \pi)$, so

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & \text{for } \theta \in [-\pi, \pi) \\ 0, & \text{else} \end{cases}$$

2 pts: f(theta)

then

2 pts: correct integration formula

$$\begin{aligned}
\mathbb{E}\left[X[n]\right] &= \mathbb{E}\left[4\cos\left(\frac{\pi}{4}n + \theta\right)\right] \\
&= \int_{-\infty}^{\infty} 4\cos(\frac{\pi}{4}n + \theta) f(\theta) d\theta \\
&= \int_{-\pi}^{\pi} 4\cos(\frac{\pi}{4}n + \theta)\frac{1}{2\pi} d\theta \\
&= \left[\frac{2}{\pi}\sin\left(\frac{\pi}{4}n + \theta\right)\right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

1pt: final answer

3 pts

(c) **(12 points)** Find the autocorrelation $r_{XX}[m,n]$ of the random process $X[n]$.

$$
\begin{aligned}
r_{XX}[m,n] &= \mathbb{E}\left[X[m]X[n]\right] \\
&= \mathbb{E}\left[4\cos\left(\frac{\pi}{4}m+\theta\right)4\cos\left(\frac{\pi}{4}n+\theta\right)\right] && (1)\\
&= 16\mathbb{E}\left[\frac{1}{2}\left(\cos\left(\frac{\pi}{4}(m+n)+2\theta\right)+\cos\left(\frac{\pi}{4}(m-n)\right)\right)\right] && (2)\\
&= 8\int_{-\infty}^{\infty}\cos\left(\frac{\pi}{4}(m+n)+2\theta\right)+\cos\left(\frac{\pi}{4}(m-n)\right)f(\theta)d\theta \\
&= 8\int_{-\pi}^{\pi}\cos\left(\frac{\pi}{4}(m+n)+2\theta\right)+\cos\left(\frac{\pi}{4}(m-n)\right)\frac{1}{2\pi}d\theta \\
&= \frac{4}{\pi}\int_{-\pi}^{\pi}\cos\left(\frac{\pi}{4}(m+n)+2\theta\right)+\cos\left(\frac{\pi}{4}(m-n)\right)d\theta \\
&= \frac{4}{\pi}\left[\frac{1}{2}\sin\left(\frac{\pi}{4}(m+n)+2\theta\right)+\cos\left(\frac{\pi}{4}(m-n)\right)\theta\right]_{-\pi}^{\pi} \\
&= \frac{4}{\pi}\cos\left(\frac{\pi}{4}(m-n)\right)\times 2\pi \\
&= 8\cos\left(\frac{\pi}{4}(m-n)\right)
\end{aligned}
$$

6 pts: derivation

3 pts

Note that $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y)+\cos(x-y)\right)$ is used to derive (2) from (1).

(d) **(3 points)** Is $X[n]$ a wide-sense stationary random process? Explain why or why not.

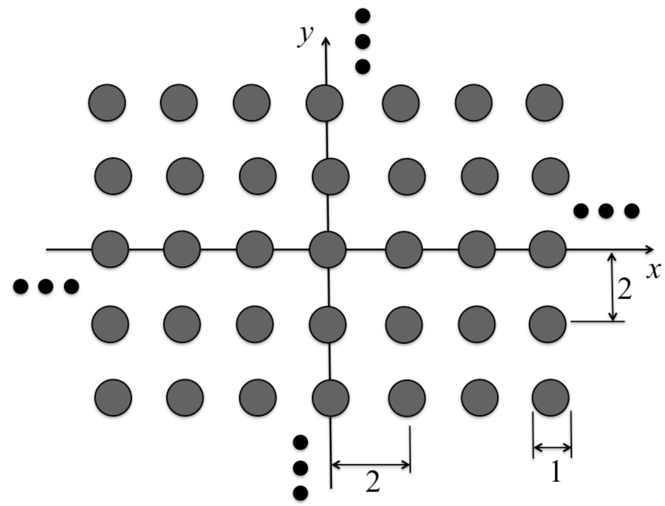
Yes. 1 pt

(a) $\mathbb{E}\left[X[n]\right] = 0$, which is a constant.

(b) $r_{XX}[m,n] = 8\cos\left(\frac{\pi}{4}(m-n)\right)$, which is only a function of the time difference.

2 pts

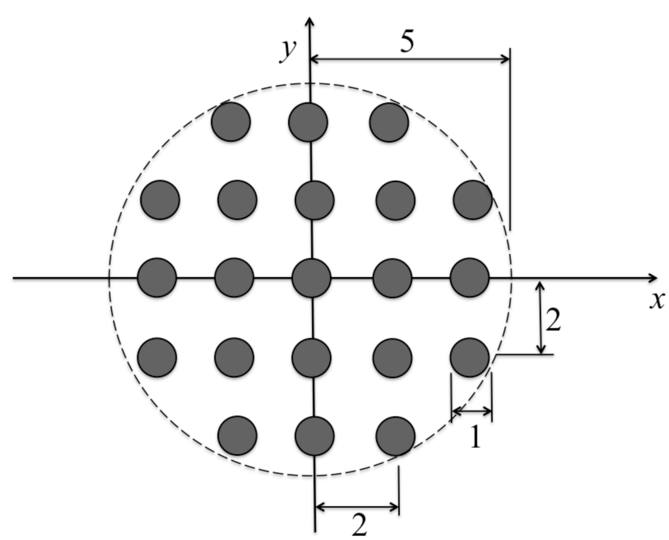
4. (25 pts) Consider the 2-D signal $f(x,y)$ shown below, which repeats infinitely far in both the x and the y directions. It has value 1 in the dark gray regions, and value 0 elsewhere.



a. (6) Find a simple expression for the CSFT $F(u,v)$ of $f(x,y)$. Your expression should not contain any operators like comb or rep. Summations are OK.

b. (6) Sketch $F(u,v)$ with sufficient detail and accuracy to indicate that you know what it looks like. Be sure to dimension all important quantities.

Now consider the 2-D signal $g(x,y)$ shown below, which is limited to a circular region centered at the origin with radius 5. It also has value 1 in the dark gray regions, and value 0 elsewhere.



c. (6) Find a simple expression for the CSFT $G(u,v)$ of $g(x,y)$. Your expression should not contain any operators like comb or rep. Summations are OK.

d. (7) Sketch $G(u,v)$ with sufficient detail and accuracy to indicate that you know what it looks like. Be sure to dimension all important quantities.

Problem 4

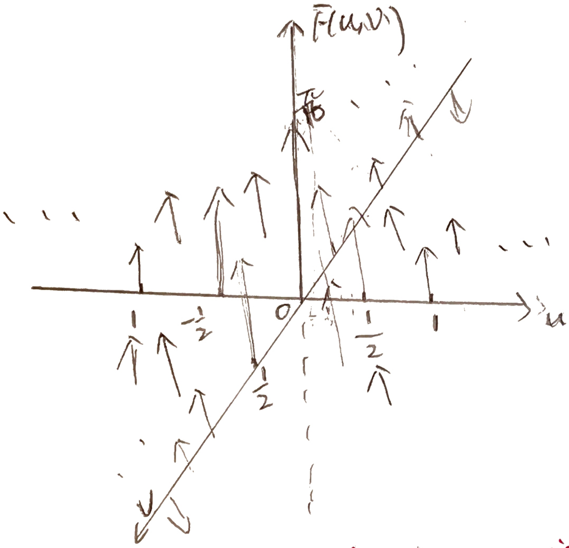
a. $f(x,y) = rep_{2,2}(circ(x,y))$ 2pt } 4pt

$F(u,v) = \frac{1}{4} comb_{\frac{1}{2},\frac{1}{2}}(jinc(u,v))$

$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} jinc(\frac{k}{2}, \frac{l}{2}) \cdot \delta(u - \frac{k}{2}, v - \frac{l}{2})$ 2pt.

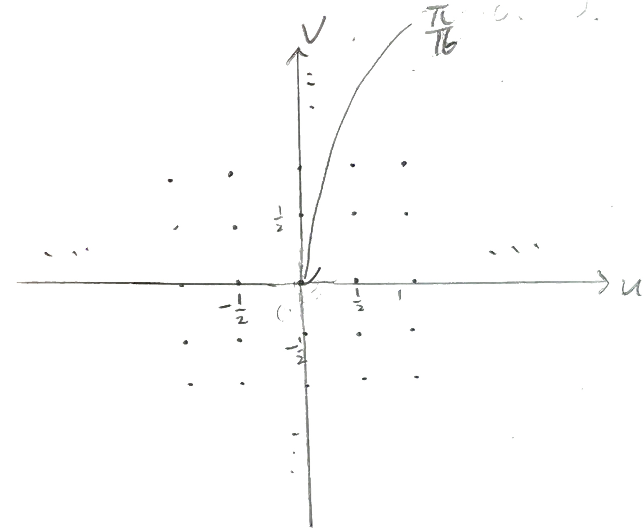
" } " means don't have to include f(x,y) or g(x,y), but can obtain partial point if got F(u,v) or G(u,v) incorrect

b.



3pt

or



1pt indicate u,v axis 1/2 separation

1pt indicate one value of the point

1pt indicate continue on u,v axis (infinite)

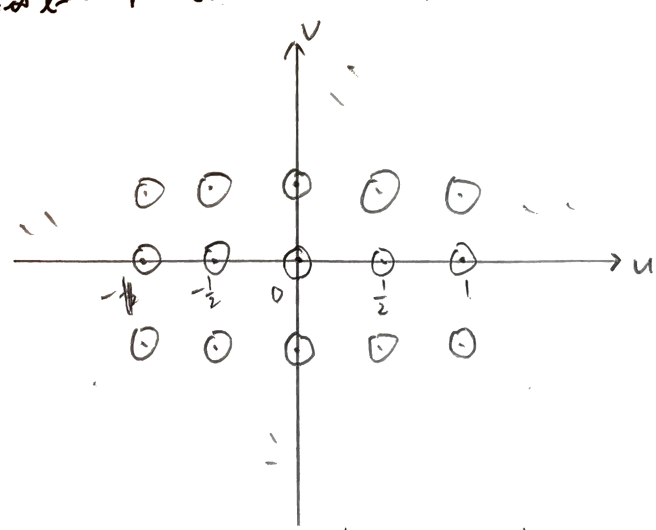
c. $g(x,y) = f(x,y) \cdot circ(\frac{x}{10}, \frac{y}{10})$ 2pt } 4pt

$G(u,v) = F(u,v) ** 10^2 jinc(10u, 10v)$ 2pt

$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} jinc(\frac{k}{2}, \frac{l}{2}) \cdot \delta(u - \frac{k}{2}, v - \frac{l}{2}) ** 100\, jinc(10u, 10v)$ 1pt } 2pt

$= 25 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} jinc(\frac{k}{2}, \frac{l}{2}) \cdot jinc(10(u - \frac{k}{2}), 10(v - \frac{k}{2}))$

d. 2pt



1pt indicate separation 1/2

1pt indicate peak value or specific point value

1pt indicate continue on u,v axis (to inf).

peak value $\frac{25\pi}{16}$ at origin

1pt → "⊙" jinc function

5. (25 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

	$h[m,n]$	$n=-1$	$n=0$	$n=1$
m	-1	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$
	0	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$
	1	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{6}$

a. (9) Find the output $g[m,n]$ when this filter is applied to the following 9×9 input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 9×9 set of pixels in the input image.

0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	1	1	1	0	0	0
0	0	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, the $\mu = \nu$, and the $\mu = -\nu$ axis.

c. (3) Using your results from parts (a) and (b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

d. (1) Is the filter DC-preserving? Why or why not?

Problem 5.

a. Output $g[m,n]$:

9 pts
2 pts for 0's
2 pts for 1's
2 pts for bottom edge
2 pts for left & right edges
1 pt for diagonal edges

0	0	0	-1/6	2/3	-1/6	0	0	0
0	0	-1/6	1/3	1	1/3	-1/6	0	0
0	-1/6	1/3	2/3	4/3	2/3	1/3	-1/6	0
-1/6	1/3	2/3	7/6	1	7/6	2/3	1/3	-1/6
-1/3	5/6	7/6	1	1	1	7/6	5/6	-1/3
-1/2	3/2	1	1	1	1	1	3/2	-1/2
-1/2	3/2	1	1	1	1	1	3/2	-1/2
-1/3	1	2/3	2/3	2/3	2/3	2/3	1	-1/3
-1/6	1/2	1/3	1/3	1/3	1/3	1/3	1/2	-1/6

b. Using separability:

$$h[m,n] = h[m] \times h[n]$$

$$h_1[m] = \delta[m-1] + \delta[m] + \delta[m+1]$$

$$h_2[n] = -\frac{1}{6}\delta[n-1] + \frac{2}{3}\delta[n] - \frac{1}{6}\delta[n+1]$$

2 pts

$$H_1(\mu) = 1 + 2cos(\mu)$$

$$H_2(\nu) = \frac{1}{6}(4 - 2cos(\nu))$$

2 pts

$$H(\mu,\nu) = H_1(\mu) \times H_2(\nu)$$

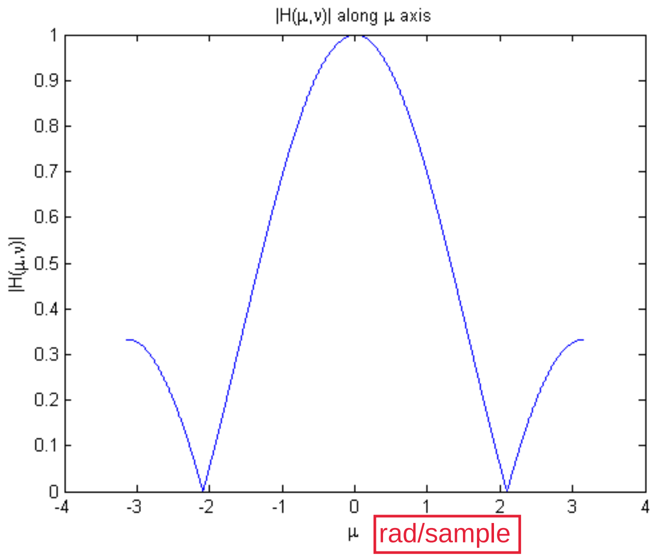
$$= (1 + 2cos(\mu))\frac{1}{6}(4 - 2cos(\nu))$$

1 pts

$$= \frac{1}{3}(2 + 4cos(\mu) - cos(\nu) - 2cos(\mu)cos(\nu))$$

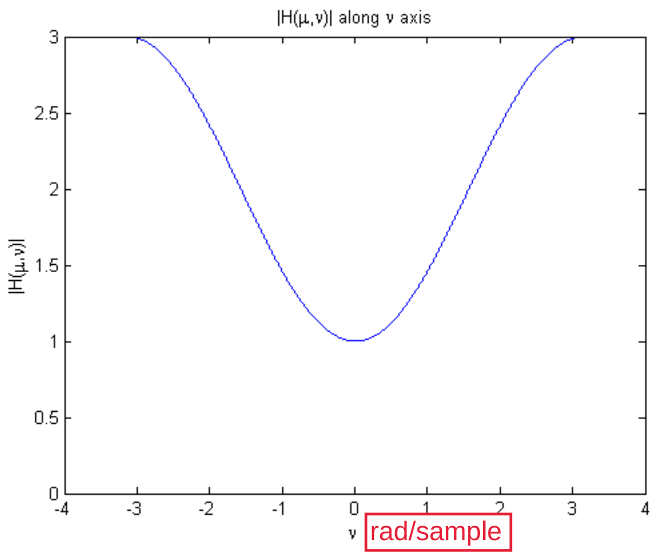
Plots:

$$H(\mu, 0) = \frac{1}{3}(1 + 2cos(\mu))$$



2 pts

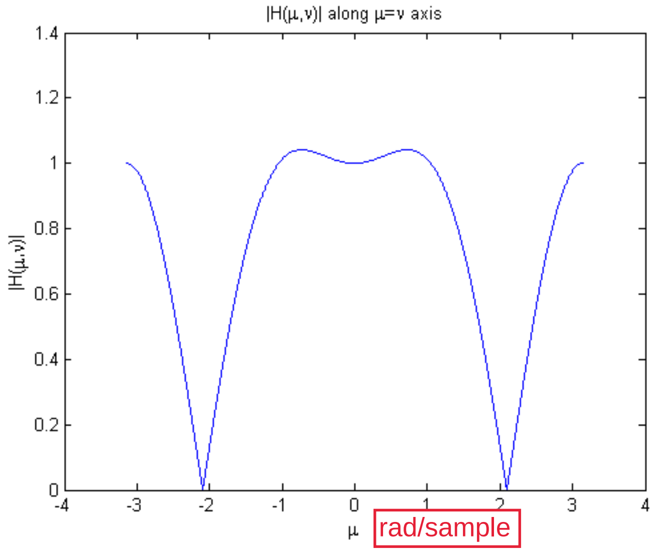
$$H(0, \nu) = 2 - cos(\nu)$$



2 pts

Plots:

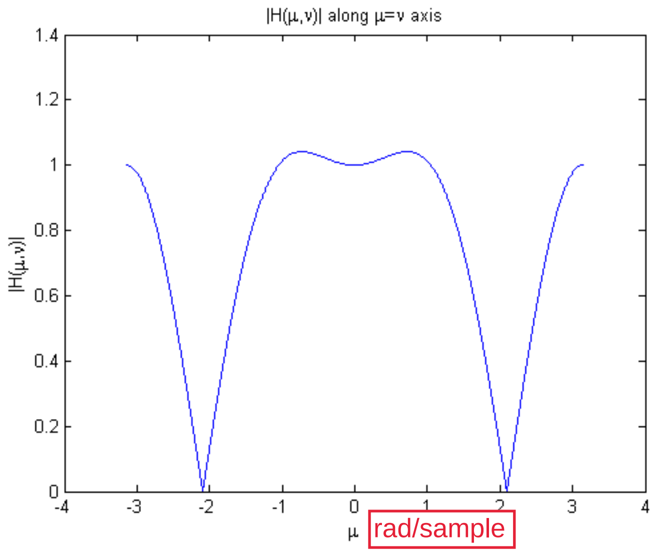
$$H(\mu,\mu) = cos(\mu) + \frac{1}{3}(1 - cos(2\mu))$$



2 pts

$$H(\mu,-\mu) = cos(\mu) + \frac{1}{3}(1 - cos(2\mu)) = H(\mu,\mu)$$

Same sketch as $H(\mu,\nu)$ along the $\mu = \nu$ axis.



1 pts

c. $H(\mu, \nu)$ along μ axis is a low-pass filter with a slow roll-off. This filter smoothes the edges in the vertical direction. This can be seen at the bottom portion of the filtered image. 1 pts
$H(\mu, \nu)$ along ν axis boosts high frequencies. This filter enhances the edges in the horizontal direction. This can be seen at the left and right portions of the filtered image. 1 pts
Along the $\mu = \nu$ axis the filter suppresses the frequencies close to $2\pi/3$. Low frequencies and $\mu = \pi$ are mostly unaffected by this filter. The diagonal edges are preserved. The center of the region of 1's is preserved because this filter is DC-preserving. 1 pts

d. Yes. This filter is DC-preserving.

$$\sum_m \sum_n h[m, n] = 1$$

1 pts

1. (20)	_______
2. (30)	_______
3. (25)	_______
4. (25)	_______
5. (25)	_______
Total (125)	_______