24 April 2020 **Name:___________________**

**ECE 438** **Exam No. 3 - Revised** **Spring 2020**

- You have 48 hours to work the following four problems. So you should e-mail a scan or images of your solution to ece438@ecn.purdue.edu by 2:30p EDT on Friday 24 April.
- Be sure to show all your work to obtain full credit.
- The exam is open book and open notes.
- Please do **NOT** discuss the problems with anyone else.

1. (20 pts.) Consider two random variables $X$ and $Y$ which are jointly distributed according to the following bivariate density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}(x+y), & 0 \le x \le 1, \text{ and } 0 \le y \le 1, \\ \frac{1}{2}(-x-y), & -1 \le x \le 0, \text{ and } -1 \le y \le 0 \\ 0, & \text{else} \end{cases}$$

a. (3) Sketch $f_{XY}(x,y)$.

b. (4) Find and sketch the marginal densities $f_X(x)$ and $f_Y(y)$.

c. (1) Are $X$ and $Y$ independent?

d. (7) Find the mean and variance of $X$ and $Y$ and the correlation coefficient $\rho_{XY}$ between them.

Now, suppose that we define two new random variables $U$ and $V$ according to

$$U = X + Y$$
$$V = X - Y$$

e. (5) Find the mean and variance of $U$ and $V$ and the correlation coefficient $\rho_{UV}$ between them.

# Problem 1

**(20 points)** Consider two random variables $X$ and $Y$ which are jointly distributed according to the following bivariate density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}(x+y), & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ \frac{1}{2}(-x-y), & -1 \le x \le 0 \text{ and } -1 \le y \le 0 \\ 0, & \text{else} \end{cases}$$

(a) **(3 points)** Sketch $f_{XY}(x,y)$.

(b) **(4 points)** Find and sketch the marginal densities $f_X(x)$ and $f_Y(y)$.

(c) **(1 points)** Are $X$ and $Y$ independent?

(d) **(7 points)** Find the mean and variance of $X$ and $Y$ and the correlation coefficient $\rho_{XY}$ between them.

Now, suppose that we define two new random variables $U$ and $V$ according to

$$U = X + Y$$
$$V = X - Y$$

(e) **(5 points)** Find the mean and variance of $U$ and $V$ and the correlation coefficient $\rho_{UV}$ between them.

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(a)

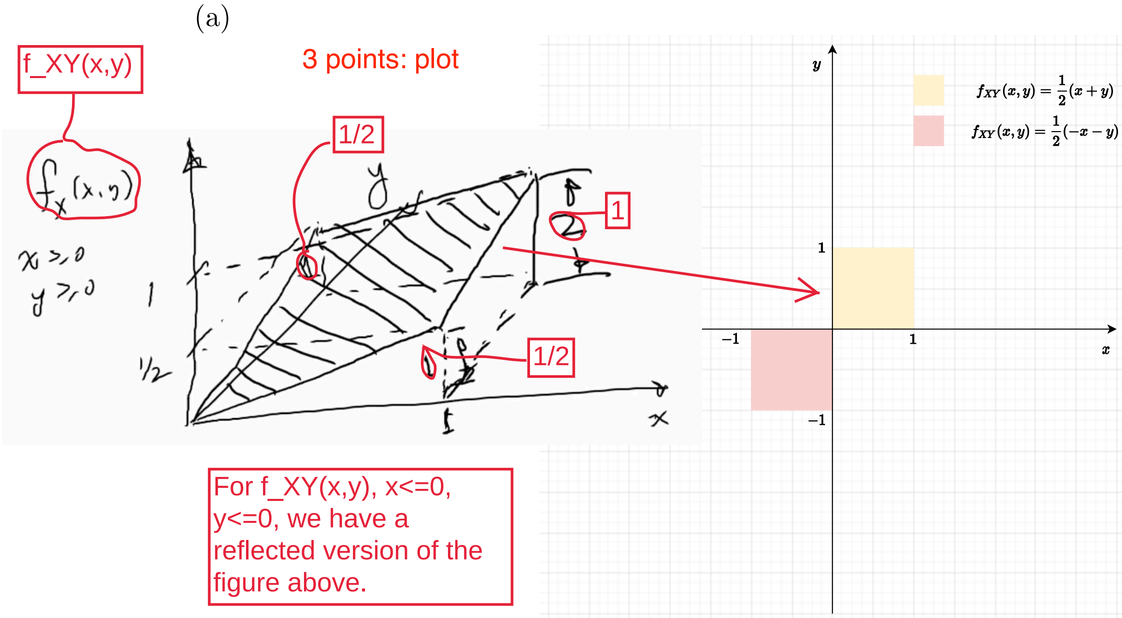



**Figure 1:** Plot of $f_{XY}(x,y)$

(b) When $-1 \le x \le 0$,

1 point: f_X(x)
1 point: f_Y(y)
1 point: plot of f_X(x)
1 point: plot of f_Y(y)

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\
&= \int_{-1}^{0} \frac{1}{2}(-x-y)dy \\
&= \frac{1}{2}\left[-xy - \frac{1}{2}y^2\right]_{-1}^{0} \\
&= -\frac{1}{2}x + \frac{1}{4}
\end{aligned}
$$

When $0 \le x \le 1$,

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\
&= \int_{0}^{1} \frac{1}{2}(x+y)dx \\
&= \frac{1}{2}\left[xy + \frac{1}{2}y^2\right]_{0}^{1} \\
&= \frac{1}{2}x + \frac{1}{4}
\end{aligned}
$$

The marginal density function of $f_Y(y)$ can be derived using the same method, so

$$
f_X(x) = \begin{cases} -\frac{1}{2}x + \frac{1}{4}, & -1 \le x \le 0 \\ \frac{1}{2}x + \frac{1}{4}, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}
$$

$$
f_Y(y) = \begin{cases} -\frac{1}{2}y + \frac{1}{4}, & -1 \le y \le 0 \\ \frac{1}{2}y + \frac{1}{4}, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}
$$

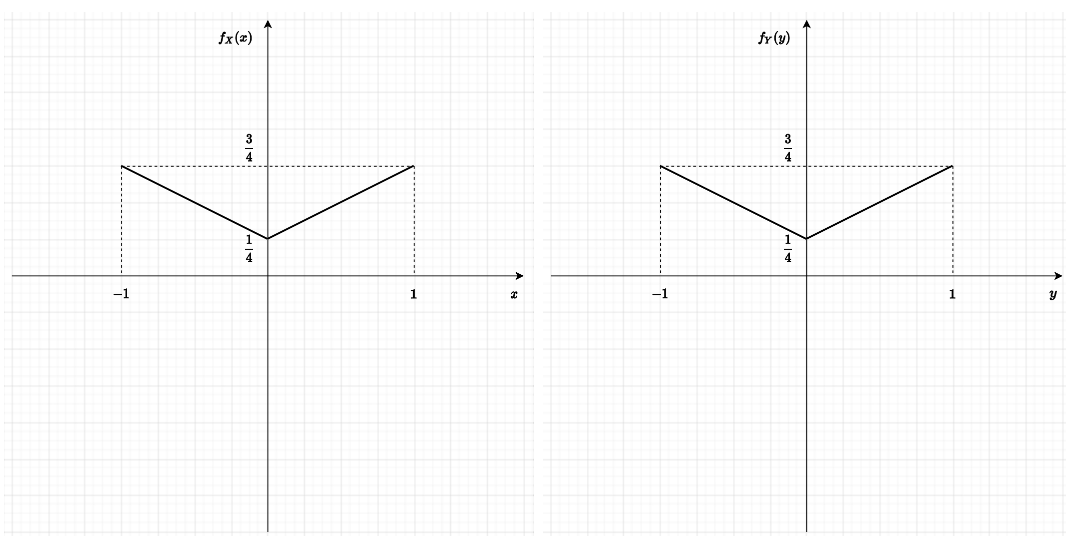



**Figure 2: Left:** $f_X(x)$ **Right:** $f_Y(y)$

(c) $X$ and $Y$ are not independent, since

1 point: not independent

$$f_X(x)f_Y(y) \neq f_{XY}(x,y)$$

(d) Calculate the mean and variance of $X$,

1 point: E[X]
1 point: Var[X]
1 point: E[Y]
1 point: Var[Y]
2 points: E[XY]
1 point: rho_{XY}

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-1}^{0} x\left(-\frac{1}{2}x+\frac{1}{4}\right)dx + \int_{0}^{1} x\left(\frac{1}{2}x+\frac{1}{4}\right)dx \\
&= 0 \\
\mathbb{E}\left[X^2\right] &= \int_{-1}^{0} x^2\left(-\frac{1}{2}x+\frac{1}{4}\right)dx + \int_{0}^{1} x^2\left(\frac{1}{2}x+\frac{1}{4}\right)dx \\
&= \frac{5}{12} \\
\mathrm{Var}[X] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \frac{5}{12}
\end{aligned}$$

Similarly, calculate the mean and variance of $Y$,

$$\begin{aligned}
\mathbb{E}[Y] &= 0 \\
\mathrm{Var}[Y] &= \frac{5}{12}
\end{aligned}$$

Calculate the correlation coefficient between them,

$$\begin{aligned}
\mathbb{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy \\
&= \int_{-1}^{0}\int_{-1}^{0} \frac{1}{2}(-x-y)xydxdy + \int_{0}^{1}\int_{0}^{1} \frac{1}{2}(x+y)xydxdy \\
&= \frac{1}{3} \\
\rho_{XY} &= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]}{\sigma_X\sigma_Y} \\
&= \frac{\frac{1}{3}-0}{\sqrt{\frac{5}{12}}\sqrt{\frac{5}{12}}} \\
&= \frac{4}{5}
\end{aligned}$$

(e) Calculate the mean and variance of $U$,

1 point: E[U]
1 point: Var[U]
1 point: E[V]
1 point: Var[V]
1 point: rho_{UV}

$$\begin{aligned}
\mathbb{E}[U] &= \mathbb{E}[X+Y] \\
&= \mathbb{E}[X] + \mathbb{E}[Y] \\
&= 0 \\
\mathrm{Var}[U] &= \mathrm{Var}[X+Y] \\
&= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]\right) \\
&= \frac{5}{12} + \frac{5}{12} + 2\left(\frac{1}{3}-0\right) \\
&= \frac{3}{2}
\end{aligned}$$

Calculate the mean and variance of $V$,

$$\begin{aligned}
\mathbb{E}\left[V\right] &= \mathbb{E}\left[X - Y\right] \\
&= \mathbb{E}\left[X\right] - \mathbb{E}\left[Y\right] \\
&= 0 \\
\text{Var}[V] &= \text{Var}[X - Y] \\
&= \text{Var}[X] + \text{Var}[Y] - 2\left(\mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]\right) \\
&= \frac{5}{12} + \frac{5}{12} - 2\left(\frac{1}{3} - 0\right) \\
&= \frac{1}{6}
\end{aligned}$$

Calculate the correlation coefficient beteen them,

$$\begin{aligned}
\mathbb{E}\left[UV\right] &= \mathbb{E}\left[(X+Y)(X-Y)\right] \\
&= \mathbb{E}\left[X^2 - Y^2\right] \\
&= \mathbb{E}\left[X^2\right] - \mathbb{E}\left[Y^2\right] \\
&= 0 \\
\rho_{UV} &= \frac{\mathbb{E}\left[UV\right] - \mathbb{E}\left[U\right]\mathbb{E}\left[V\right]}{\sigma_U \sigma_V} \\
&= \frac{0 - 0}{\sqrt{\frac{3}{2}}\sqrt{\frac{1}{6}}} \\
&= 0
\end{aligned}$$

2. (25) (Before working this problem, you may want to review the solution to Problem 3(b) on Homework No. 8, which I have just revised and re-posted. Consider a voiced phoneme for which the time-domain, continuous-time vocal tract response $v(t)$ is given by

$$v(t) = \cos\left(2\pi(1000)t\right)\text{rect}\left(\frac{t-2.5\times10^{-3}}{5\times10^{-3}}\right).$$

Assume that the pitch frequency for the speaker is 100 Hz, and that the excitation consists of a train of ideal impulses.

a. (2) Carefully sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

b. (7) Find an exact expression for the CTFT $S(f)$ of the speech waveform.

c. (2) Carefully sketch the CTFT $S(f)$. Be sure to dimension all important quantities in $S(f)$.

Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length of 50 samples. So we have

$$\tilde{S}(\omega,n) = \sum_{k=-\infty}^{\infty} s[k]w[n-k]e^{-j\omega k}$$

where

$$w[k] = \begin{cases} 1, & 0 \le k \le 49, \\ 0, & \text{else} \end{cases}$$

d. (2) Compute an exact expression for the STDTFT at time $n = 0$, i.e. $\tilde{S}(\omega,0)$.

e. (7) Compute an exact expression for the STDTFT at time $n = 50$, i.e. $\tilde{S}(\omega,50)$.

f. (2) Carefully sketch $\tilde{S}(\omega,50)$. Be sure to dimension all important quantities in $\tilde{S}(\omega,50)$. You may ignore any phase factors in your sketch.

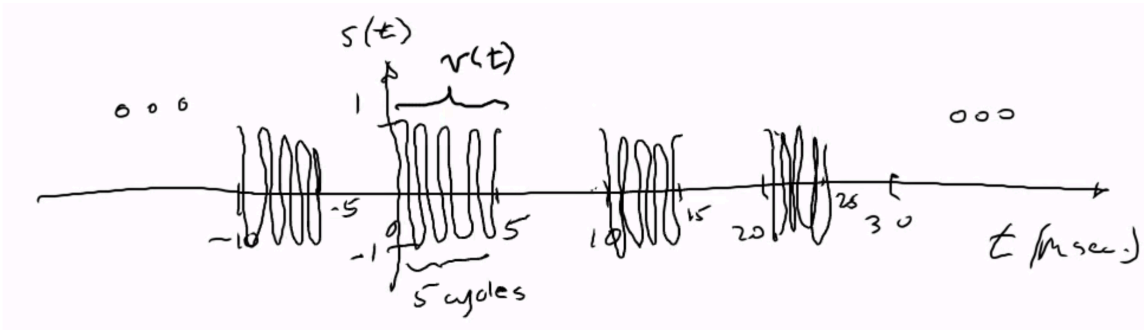
g. (3) Based on your answers to parts (e) and (f) above, approximately sketch the resulting spectrogram $\left|\tilde{S}(\omega,n)\right|$ as a function of the discrete-time index $n$ and digital frequency $\omega$ (radians/sample). Be sure to dimension all important quantities in your plot. Is this a wide-band or a narrow-band spectrogram?

# Solution for Exam 3 Problem 2 

(a) (2) Since the pitch frequency is 100 Hz, the pitch period is $P = 0.01$ sec. Thus we have that

$$\begin{aligned} s(t) &= \text{rep}_P\left\{v(t)\right\} \\ &= \sum_{k=-\infty}^{\infty} v(t - 0.01k) \end{aligned}, \tag{1}$$

which looks like




(b) From the first line of Eq. (1), we have that

$$\begin{aligned} S(f) &= \frac{1}{P}\text{comb}_{\frac{1}{P}}\left\{V(f)\right\} \\ &= \sum_{k=-\infty}^{\infty} V\left(\frac{k}{P}\right)\delta\left(f - \frac{k}{P}\right) \end{aligned}. \tag{2}$$

But

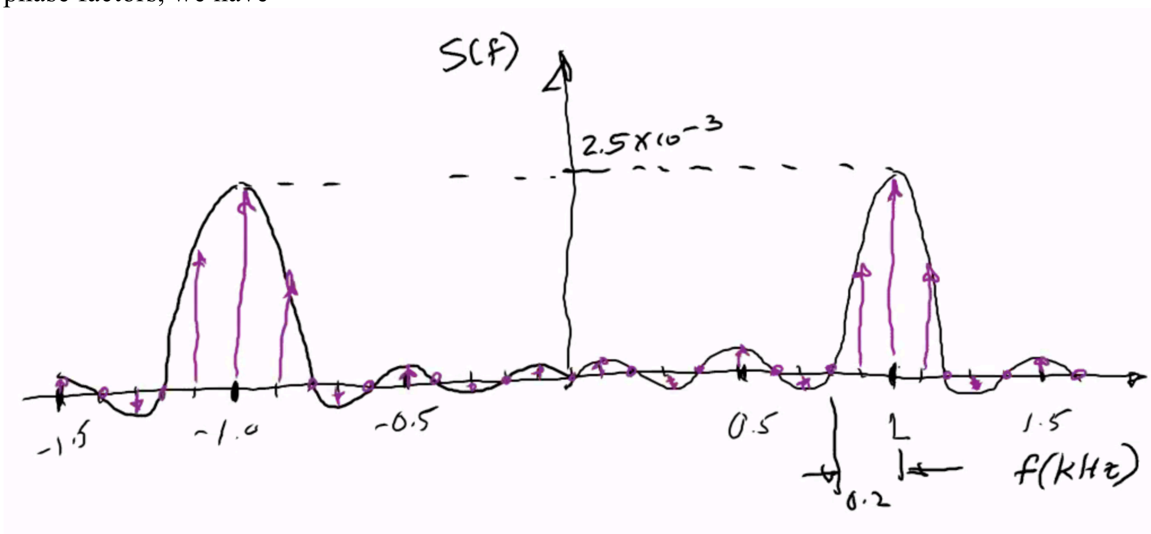
$$\begin{aligned} V(f) &= \text{CTFT}\left\{\cos\left(2\pi(1000)t\right)\text{rect}\left(\frac{t - 2.5\times 10^{-3}}{5\times 10^{-3}}\right)\right\} \\ &= 5\times 10^{-3}\left(\text{sinc}(5\times 10^{-3} f)e^{-j2\pi(2.5\times 10^{-3} f)}\right) * \frac{1}{2}\left(\delta(f - 1000) + \delta(f + 1000)\right) \\ &= 2.5\times 10^{-3}\left\{\text{sinc}(5\times 10^{-3}(f - 1000))e^{-j2\pi(2.5\times 10^{-3}(f-1000))}\right. \\ &\qquad \left. + \text{sinc}(5\times 10^{-3}(f + 1000))e^{-j2\pi(2.5\times 10^{-3}(f+1000))}\right\} \end{aligned}. \tag{3}$$

Substituting the second third and fourth lines of Eq. (3) into the second line of Eq. (2), we obtain

$$\begin{aligned} S(f) &= 2.5\times 10^{-3}\sum_{k=-\infty}^{\infty}\left\{\text{sinc}(5\times 10^{-3}(100k - 1000))e^{-j2\pi(2.5\times 10^{-3}(100k-1000))}\right. \\ &\qquad \left. + \text{sinc}(5\times 10^{-3}(100k + 1000))e^{-j2\pi(2.5\times 10^{-3}(100k+1000))}\right\}\delta\left(f - 100k\right) \\ &= 2.5\times 10^{-3}\sum_{k=-\infty}^{\infty}\left\{\text{sinc}(0.5(k - 10))e^{-j2\pi(0.25(k-10))}\right. \\ &\qquad \left. + \text{sinc}(0.5(k + 10))e^{-j2\pi(0.25(k+10))}\right\}\delta\left(f - 100k\right) \\ &= 2.5\times 10^{-3}\sum_{k=-\infty}^{\infty}\left\{\text{sinc}\left(\frac{k - 10}{2}\right)e^{-j2\pi(0.25(k-10))}\right. \\ &\qquad \left. + \text{sinc}\left(\frac{k + 10}{2}\right)e^{-j2\pi(0.25(k+10))}\right\}\delta\left(f - 100k\right) \end{aligned}, \tag{4}$$

where we have used the fact that $\frac{1}{P} = 100$.

(c) (2) Now, we are ready to sketch $S(f)$, based on the last two lines of Eq. (4). Ignoring the phase factors, we have




(d) (2) We have

$$\tilde{S}(\omega, n) = \sum_{k=-\infty}^{\infty} s[k]w[n-k]e^{-j\omega k}, \tag{5}$$

where

$$w[k] = \begin{cases} 1, & 0 \le k \le 49, \\ 0, & \text{else} \end{cases}. \tag{6}$$

Since we are sampling at a 10 kHz rate,

$$\begin{aligned} s[k] &= s(k/10^4) \\ &= \begin{cases} \cos(2\pi(1000)k/10^4), & 0 \le k \le 49 \\ 0, & -49 \le k < 1 \end{cases}. \\ &= \begin{cases} \cos(2\pi k/10), & 0 \le k \le 49 \\ 0, & -49 \le k < 1 \end{cases} \end{aligned} \tag{7}$$

Note that we are assuming here that $\text{rect}(-0.5) = 1$. Also, for $n = 0$, we have $w[-k] = \begin{cases} 1, & -49 \le k \le 0 \\ 0, & else \end{cases}$. So there is only one point of overlap at $k = 0$; and $s[k]w[-k] = \delta[k]$; and $\tilde{S}(\omega, 0) \equiv 1$. Alternatively, we could assume that $\text{rect}(-0.5) = 0$. Then,

$$s[k] = \begin{cases} \cos(2\pi k/10), & 0 < k \le 49 \\ 0, & -49 \le k \le 0 \end{cases}; \tag{8}$$

and $s[k]w[-k] \equiv 0$. Thus, $\tilde{S}(\omega,0) \equiv 0$. Either solution is acceptable.

(e) (7) For $n = 50$, we have

$$\begin{aligned} w[50-k] &= w[-(-k-50)] \\ &= \begin{cases} 1, & 0 < k \le 49 \\ 0, & else \end{cases}. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} \tilde{S}(\omega,50) &= \sum_{k=1}^{49} s[k]e^{-j\omega k} \\ &= \sum_{k=0}^{49} s[k]e^{-j\omega k} - s[0] \end{aligned}, \tag{10}$$

where in the second line of Eq. (10), we have added the $k = 0$ term to the summation, and then subtracted it from the overall result for mathematical convenience. However, if we again assume that $\text{rect}(-0.5) = 0$, then $s[0] = 0$. We will, indeed, assume that $s[0] = 0$ for the remainder of this problem. Then, we can write that

$$\tilde{S}(\omega,50) = \sum_{k=-\infty}^{\infty} s'[k]w'[k]e^{-j\omega k}, \tag{11}$$

where $s'[k] = \cos(2\pi(k/10))$ and $w'[k] = \begin{cases} 1, & 0 \le k \le 49 \\ 0, & else \end{cases}$. From the product theorem for the DTFT, we have that

$$\tilde{S}(\omega,50) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S'(\omega-\mu)W'(\mu)d\mu, \tag{12}$$

So we have

$$S'(\omega) = \pi\left\{\delta(\omega-\pi/5) + \delta(\omega+\pi/5)\right\}, -\pi \le \omega \le \pi, \tag{13}$$

and

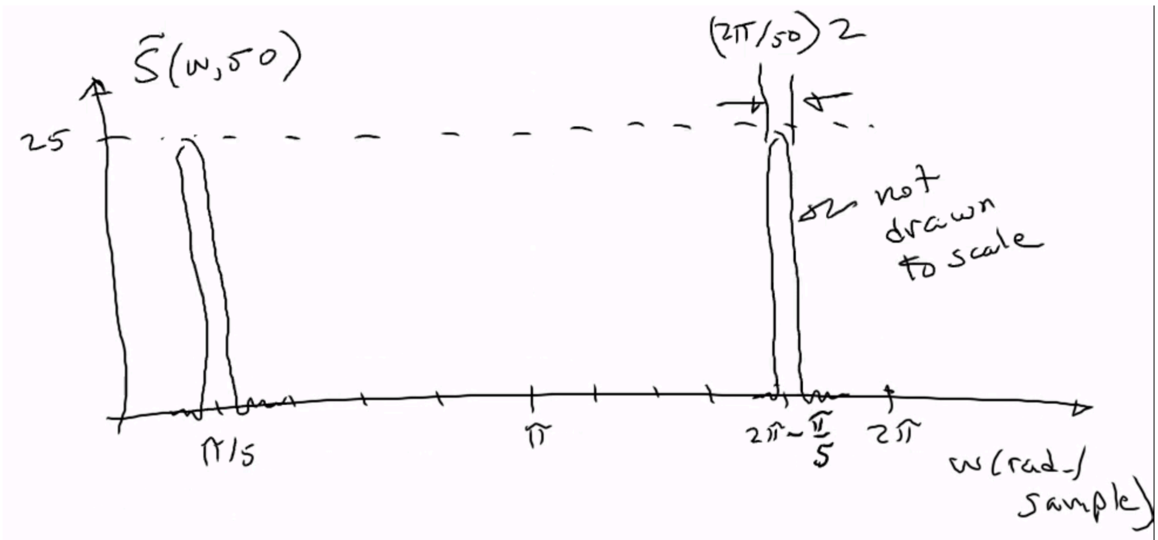
$$W'(\omega) = \text{psinc}_{50}\left(\omega\right)e^{-j\omega(49/2)}. \tag{14}$$

Using the fact that convolution of any function with an impulse, shifts that function to the location of the impulse, we obtain

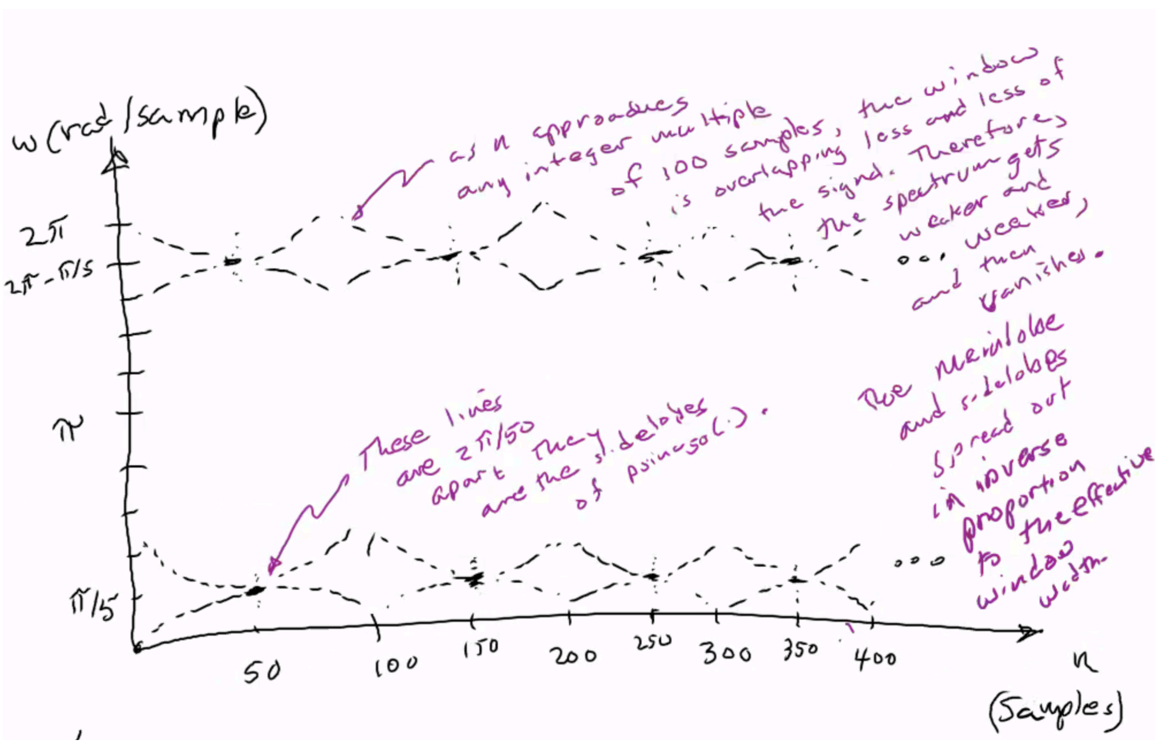
$$\begin{aligned}\tilde{S}(\omega,50) &= \frac{1}{2}\left\{\text{psinc}_{50}(\omega-\pi/5)e^{-j(\omega-\pi/5)(49/2)} + \text{psinc}_{50}(\omega+\pi/5)e^{-j(\omega+\pi/5)(49/2)}\right\} \\ &= \frac{1}{2}\left\{\text{psinc}_{50}(\omega-\pi/5)e^{-j(\omega-\pi/5)(49/2)} + \text{psinc}_{50}(\omega-(2\pi-\pi/5))e^{-j(\omega(2\pi-\pi/5))(49/2)}\right\}\end{aligned} \quad . \quad (15)$$

Here, the second line of Eq. (15) follows from the fact that the DTFT is always periodic with period $2\pi$ .

(f) (2)



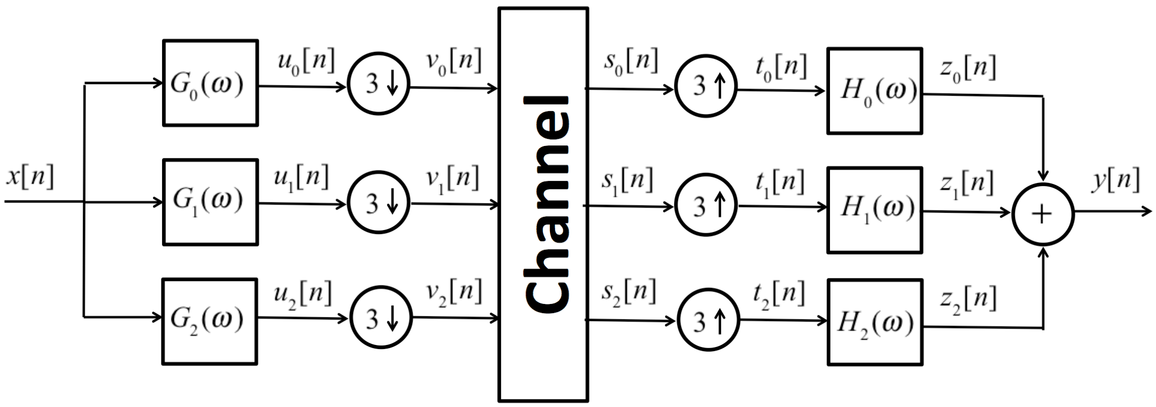

(g) (3)



## Solution for Exam 3 Problem 2 5

This is a wideband spectrogram, since the pitch period is 0.01 sec., while the window duration is 50 samples, or 0.005 sec. Thus, we can resolve the spectrum $V(\omega)$ during each repetition of the vocal tract response $v[n]$. There is only one formant frequency at 1 kHz, which corresponds to the digital frequencies $\pi / 5$ and $2\pi - \pi / 5$ rad./sample.

3. (35) (Before working this problem, you may want to download and read the document "Analysis of 2-Channel Filter Bank" that is posted at the course website as Module 4.2.5. If you have downloaded it previously, please download it again, since one of the figures has been corrected.) Consider the three-channel filter bank shown below:




We will consider the channel to be ideal; so $s_i[n] \equiv v_i[n], i = 0,1,2$ .

a. (8) Find necessary and sufficient conditions that must be satisfied by the three analysis filters $G_i(\omega), i = 0,1,2$ and the three synthesis filters $H_i(\omega), i = 0,1,2$ in order that we have perfect reconstruction, i.e. $y[n] \equiv x[n]$ .

Suppose that $H_i(\omega) \equiv \frac{1}{3} G_i(\omega), i = 0,1,2$ and

$$G_i(\omega) = 3 \cdot \text{rect}\left( \frac{\omega - (2\pi/3)i}{2\pi/3} \right), -\pi/3 \le \omega \le 5\pi/3, i = 0,1,2$$

b. (3) Carefully sketch $G_i(\omega), 0 \le \omega \le 2\pi, i = 0,1,2$ , being sure to dimension all important quantities. (Note that all discrete frequency domain quantities must be periodic in $\omega$ with period $2\pi$ .

Suppose that $X(\omega) = (1 - |\omega|/\pi), |\omega| < \pi$ .

c. (3) Carefully sketch $X(\omega)$ , being sure to dimension all important quantities.

d. (3) According to the diagram above, carefully sketch $U_i(\omega), i = 0,1,2$ , being sure to dimension all important quantities.

e. (3) According to the diagram above, carefully sketch $V_i(\omega), i = 0,1,2$ , being sure to dimension all important quantities.

f. (3) According to the diagram above, carefully sketch $T_i(\omega), i = 0,1,2$ , being sure to dimension all important quantities.

g. (3) According to the diagram above, carefully sketch $Z_i(\omega), i = 0,1,2$ , being sure to dimension all important quantities.

h. (3) Finally, according to the diagram above, carefully sketch $Y(\omega)$ , being sure to dimension all important quantities.

i. (6) Show that the filters $H_i(\omega) \equiv \frac{1}{3} G_i(\omega), i = 0,1,2$ , for which you sketched $G_i(\omega), 0 \le \omega \le 2\pi, i = 0,1,2$ , in part (b) above satisfy the conditions that you derived for perfect reconstruction in part (a) above.

(a)

Following the same process as performed in Homework 8 problem 5a, let us look at the frequency response of an arbitrary channel, $i, i = 0,1,2$.

Recall:

1 pt. for each correct expression for frequency response for signals u_i, v_i, t_i, z_i, and y (5pts. total)

Downsample: $Y(\omega) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(\frac{\omega-2\pi k}{D}\right)$

Upsample: $Y(\omega) = X(L\omega)$

$$U_i(\omega) = X(\omega)G_i(\omega)$$

$$V_i(\omega) = \frac{1}{3}\sum_{k=0}^{2} U_{i}\left(\frac{\omega-2\pi k}{3}\right) = \frac{1}{3}\sum_{k=0}^{2} X\left(\frac{\omega-2\pi k}{3}\right)G_i\left(\frac{\omega-2\pi k}{3}\right)$$

$$S_i(\omega) = V_i(\omega)$$

$$T_i(\omega) = S_i(3\omega) = \frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)G_i\left(\omega - \frac{2\pi}{3}k\right)$$

$$Z_i(\omega) = T_i(\omega)H_i(\omega) = \frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)G_i\left(\omega - \frac{2\pi}{3}k\right)H_i(\omega)$$

For perfect reconstruction, which means $y[n] = x[n]$, implying

$$Y(\omega) = X(\omega)$$

$$\text{Where } Y(\omega) = Z_0(\omega) + Z_1(\omega) + Z_2(\omega)$$

We have

$$Z_0(\omega) + Z_1(\omega) + Z_2(\omega) = \frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)G_0\left(\omega - \frac{2\pi}{3}k\right)H_0(\omega)$$

$$+\frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)G_1\left(\omega - \frac{2\pi}{3}k\right)H_1(\omega) + \frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)G_2\left(\omega - \frac{2\pi}{3}k\right)H_2(\omega)$$

$$= \frac{1}{3}\sum_{k=0}^{2} X\left(\omega - \frac{2\pi}{3}k\right)\left\{G_0\left(\omega - \frac{2\pi}{3}k\right)H_0(\omega) + G_1\left(\omega - \frac{2\pi}{3}k\right)H_1(\omega) + G_2\left(\omega - \frac{2\pi}{3}k\right)H_2(\omega)\right\}$$

$$= \frac{1}{3}X(\omega)\{G_0(\omega)H_0(\omega) + G_1(\omega)H_1(\omega) + G_2(\omega)H_2(\omega)\}$$
$$+\frac{1}{3}X\left(\omega - \frac{2\pi}{3}\right)\left\{G_0\left(\omega - \frac{2\pi}{3}\right)H_0(\omega) + G_1\left(\omega - \frac{2\pi}{3}\right)H_1(\omega) + G_2\left(\omega - \frac{2\pi}{3}\right)H_2(\omega)\right\}$$
$$+\frac{1}{3}X\left(\omega - \frac{4\pi}{3}\right)\left\{G_0\left(\omega - \frac{4\pi}{3}\right)H_0(\omega) + G_1\left(\omega - \frac{4\pi}{3}\right)H_1(\omega) + G_2\left(\omega - \frac{4\pi}{3}\right)H_2(\omega)\right\}$$

3pts. Correct Solution (1pt each)

In order to satisfy $Y(\omega) = X(\omega)$, we need:

$$G_0(\omega)H_0(\omega) + G_1(\omega)H_1(\omega) + G_2(\omega)H_2(\omega) = 3$$

And

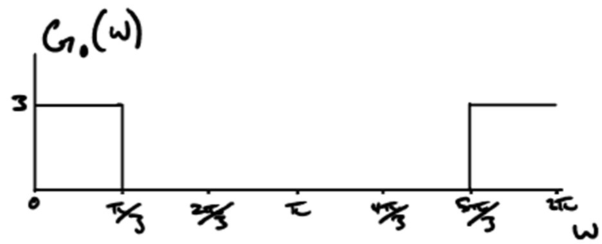
$$G_0\left(\omega - \frac{2\pi}{3}\right)H_0(\omega) + G_1\left(\omega - \frac{2\pi}{3}\right)H_1(\omega) + G_2\left(\omega - \frac{2\pi}{3}\right)H_2(\omega) = 0$$

And

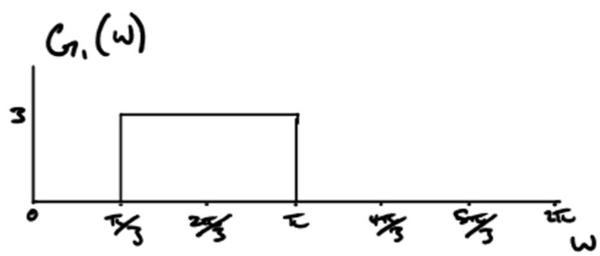
$$G_0\left(\omega - \frac{4\pi}{3}\right)H_0(\omega) + G_1\left(\omega - \frac{4\pi}{3}\right)H_1(\omega) + G_2\left(\omega - \frac{4\pi}{3}\right)H_2(\omega) = 0$$

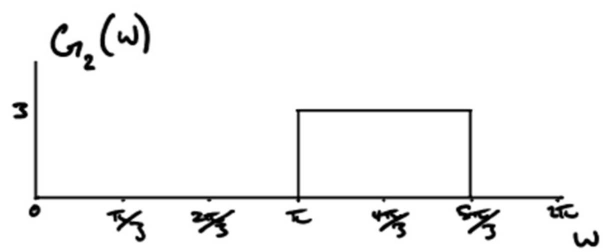
(b)

All following plots are periodic with period $2\pi$.

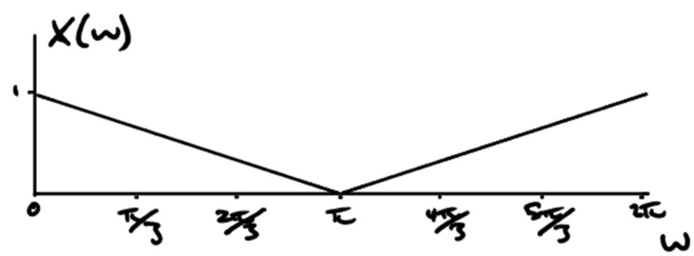



1/2 pt. For each correct G_i(w)
1/2 pt. per plot for correct axis dimensions



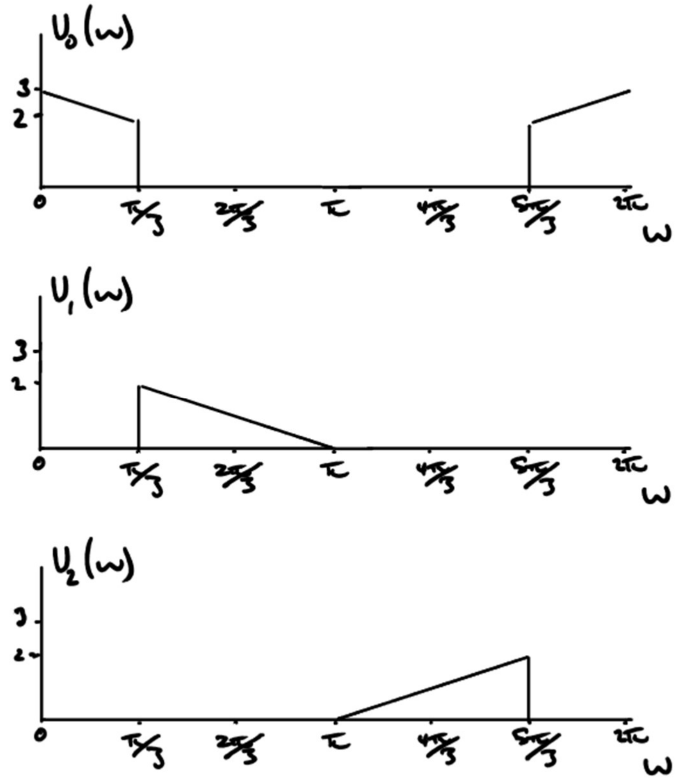




(c)



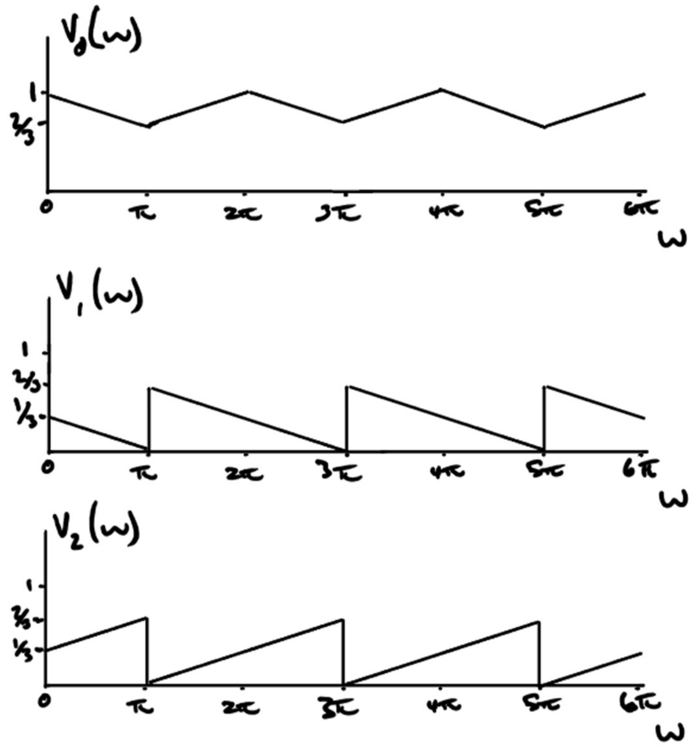

2 1/2 pts. Correct X(w)
1/2 pt. For correct axis dimesions

(d)



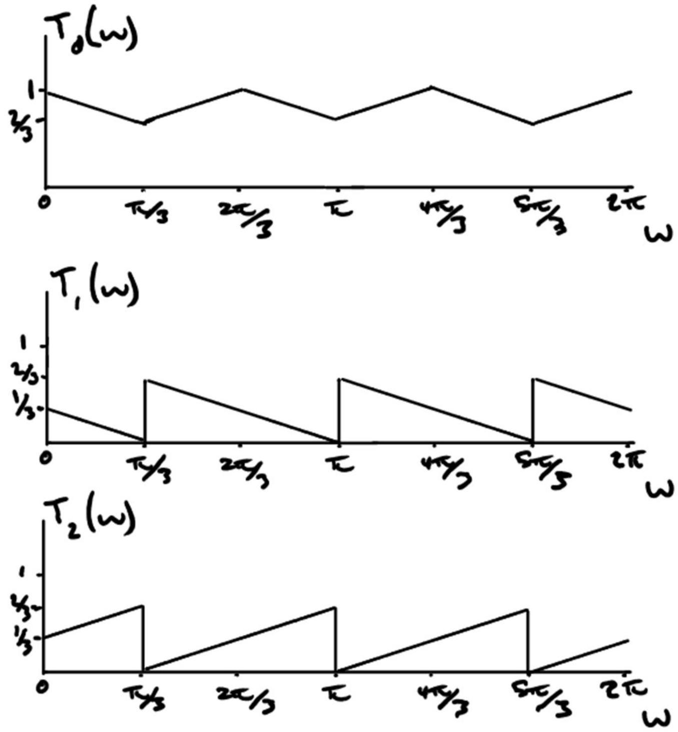

1/2 pt. For each correct U_i(w)
1/2 pt. per plot for correct axis dimensions

(e)



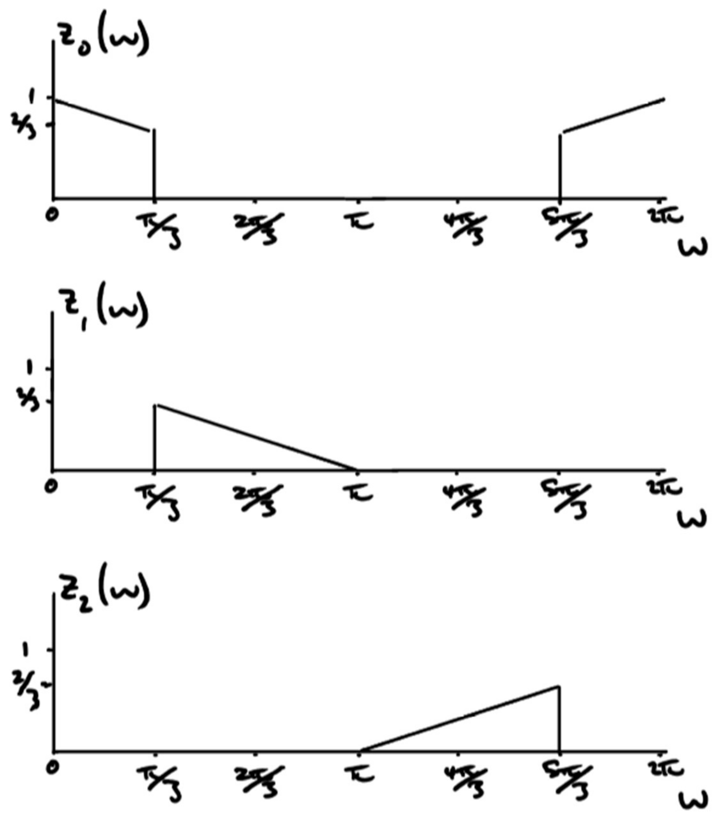

1/2 pt. For each correct V_i(w)
1/2 pt. per plot for correct axis dimensions

(f)



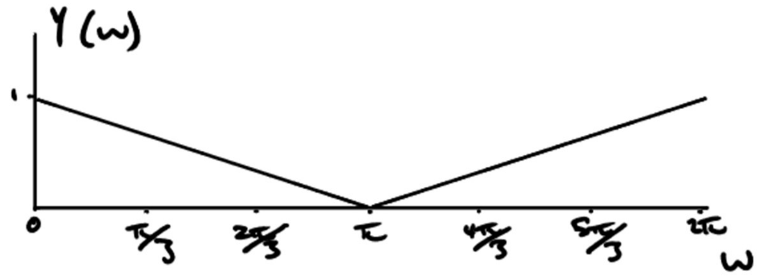

1/2 pt. For each correct T_i(w)
1/2 pt. per plot for correct axis dimensions

(g)




1/2 pt. For each correct Z_i(w)
1/2 pt. per plot for correct axis dimensions

(h)




2 1/2 pts. Correct Y(w)
1/2 pt. For correct axis dimesions

(i)

$$G_0(\omega)H_0(\omega)+G_1(\omega)H_1(\omega)+G_2(\omega)H_2(\omega)$$
$$=\left[3\,\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\right]+\left[3\,\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\right]$$
$$+\left[3\,\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\right]$$
$$=3\left[\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)+\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)+\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\right]$$
$$=3(1)=3$$

2 pts. per correctly evaluated expression (6 pts. total)

And

$$G_0\left(\omega-\frac{2\pi}{3}\right)H_0(\omega)+G_1\left(\omega-\frac{2\pi}{3}\right)H_1(\omega)+G_2\left(\omega-\frac{2\pi}{3}\right)H_2(\omega)$$
$$=\left[3\,\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\right]+\left[3\,\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\right]$$
$$+\left[3\,\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\right]$$
$$=0$$

And

$$G_0\left(\omega-\frac{4\pi}{3}\right)H_0(\omega)+G_1\left(\omega-\frac{4\pi}{3}\right)H_1(\omega)+G_2\left(\omega-\frac{4\pi}{3}\right)H_2(\omega)$$
$$=\left[3\,\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\right]+\left[3\,\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\right]$$
$$+\left[3\,\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)\right]$$
$$=0$$

Since

$$\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)=\mathrm{rect}\left(\frac{\omega}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)=\mathrm{rect}\left(\frac{\omega-2\pi/3}{2\pi/3}\right)\mathrm{rect}\left(\frac{\omega-4\pi/3}{2\pi/3}\right)=0$$

Because the rectangles do not overlap.

4. (20)

Consider the signal $x(t)$ defined as follows

$$x(t) = t^2, 0 \le t \le 1$$

We wish to approximate $x(t)$ over the interval $0 \le t \le 1$ by the signal $\hat{x}(t)$ given by

$$\hat{x}(t) = a_0 b_0(t) + a_1 b_1(t) ,$$

where the basis functions $b_0(t)$ and $b_1(t)$ are given by

$$b_0(t) = 1, 0 \le t \le 1, \text{ and } b_1(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ -1, & 1/2 < t \le 1 \end{cases} ,$$

and $a_0$ and $a_1$ are constants chosen to minimize

$$\varepsilon = \int_0^1 \left[ \hat{x}(t) - x(t) \right]^2 dt .$$

a. (10) Determine the values for $a_0$ and $a_1$ that will minimize $\varepsilon$.

b. (4) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes for the optimal values for $a_0$ and $a_1$ that you determined in part (a) above.

c. (6) Compute the mean-squared error $\varepsilon$.

## Problem 4 (20)

Consider the signal $x(t)$ defined as follows

$$x(t) = t^2, 0 \leq t \leq 1$$

We wish to approximate $x(t)$ over the interval $0 \leq t \leq 1$ by the signal $\hat{x}(t)$ given by

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and $a_0$ and $a_1$ are constants chosen to minimize

$$\varepsilon = \int_0^1 [\hat{x}(t) - x(t)]^2 dt$$

.

a. (10) Determine the values for $a_0$ and $a_1$ that will minimize $\epsilon$ .

$$\begin{aligned} \varepsilon &= \int_0^1 [\hat{x}(t) - x(t)]^2 dt \\ &= \int_0^1 [a_0 b_0(t) + a_1 b_1(t) - t^2]^2 dt \\ &= \int_0^{0.5} [a_0 + a_1 - t^2]^2 dt + \int_{0.5}^1 [a_0 - a_1 - t^2]^2 dt \end{aligned}$$

1 pt: piecewise simplify

$$\begin{aligned} \frac{\partial \varepsilon}{\partial a_0} &= 2\int_0^{0.5} (a_0 + a_1 - t^2) dt + 2\int_{0.5}^1 (a_0 - a_1 - t^2) dt \\ &= 2\int_0^1 (a_0 - t^2) dt + 2\int_0^{0.5} a_1 dt - 2\int_{0.5}^1 a_1 dt \\ &= (2a_0 t - \frac{2}{3} t^3)\Big|_0^1 \\ &= 2a_0 - \frac{2}{3} \end{aligned}$$

1 pt: first derivative
1 pt: primitive functions
1 pt: definite integral

$$
\begin{aligned}
\frac{\partial \varepsilon}{\partial a_1} &= 2\int_0^{0.5}(a_0+a_1-t^2)dt - 2\int_{0.5}^{1}(a_0-a_1-t^2)dt \\
&= 2(a_0+a_1)\Big|_0^{0.5} - \frac{2}{3}t^3\Big|_0^{0.5} - 2(a_0-a_1)\Big|_{0.5}^{1} + \frac{2}{3}t^3\Big|_{0.5}^{1} \\
&= a_0+a_1+a_1-a_0-\frac{2}{3}\times\frac{1}{8}+\frac{2}{3}\left(1-\frac{1}{8}\right) \\
&= 2a_1+\frac{1}{2}
\end{aligned}
$$

1 pt: first derivative
1 pt: primitive functions
1 pt: definite integral

let $\frac{\partial \varepsilon}{\partial a_0}=0$, $\frac{\partial \varepsilon}{\partial a_1}=0$, and get

$$a_0=\frac{1}{3},\quad a_1=-\frac{1}{4}$$

2 pts: first derivatives equal to 0

Second derivative check for minimization:

$$
\begin{aligned}
\frac{\partial \varepsilon}{\partial a_0} &= 2>0 \\
\frac{\partial \varepsilon}{\partial a_1} &= 2>0
\end{aligned}
$$

1 pt: second derivative check for convex

Therefore, $a_0=\frac{1}{3}$, $a_1=-\frac{1}{4}$ minimize $\varepsilon$.

b. (4) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes for the optimal values for $a_0$ and $a_1$ that you determined in part (a) above.

$$
\hat{x}(t)=\begin{cases} a_0+a_1=\frac{1}{12}, & 0\le t\le 1/2 \\ a_0-a_1=\frac{7}{12}, & 1/2<t\le 1 \end{cases}
$$

1 pt: hat x(t) calculation
1 pt: hat x(t) sketch (0.5pts for indicating value at t=1/2).
1 pt: x(t) sketch
1 pt: label both axes.

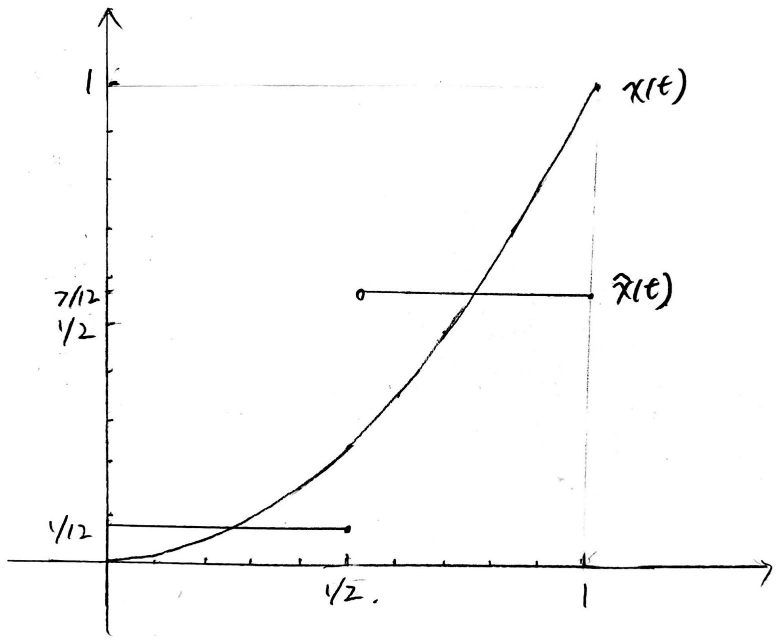


Figure 1: $x(t)$ and $\hat{x}(t)$

c. (6) Compute the mean-square error $\epsilon$

$$
\begin{aligned}
\varepsilon &= \int_0^1 (\hat{x}(t) - x(t))^2 dt \\
&= \int_0^{0.5} (a_0 + a_1 - t^2)^2 dt + \int_{0.5}^1 (a_0 - a_1 - t^2)^2 dt \\
&= \int_0^{0.5} (a_0^2 + a_1^2 + t^4 + 2a_0a_2 - 2a_0t^2 - 2a_1t^2)dt + \int_{0.5}^1 (a_0^2 + a_1^2 + t^4 - 2a_0a_2 - 2a_0t^2 + 2a_1t^2)dt \\
&= \int_0^1 (a_0^2 + a_1^2 + t^4 - 2a_0t^2)dt + \int_0^{0.5} (2a_0a_1 - 2a_1t^2)dt + \int_{0.5}^1 (-2a_0a_1 + 2a_1t^2)dt \\
&= (a_0^2 + a_1^2)t\Big|_0^1 + \frac{1}{5}t^5\Big|_0^1 - \frac{2}{3}a_0t^3\Big|_0^1 + 2a_0a_1t\Big|_0^{0.5} - \frac{2}{3}a_1t^3\Big|_0^{0.5} - 2a_0a_1t\Big|_{0.5}^1 + \frac{2}{3}a_1t^3\Big|_{0.5}^1 \\
&= a_0^2 + a_1^2 + \frac{1}{5} - \frac{2}{3}a_0 + a_0a_1 - \frac{2}{3}a_1 \times \frac{1}{8} - a_0a_1 + \frac{2}{3}a_1(1 - \frac{1}{8}) \\
&= a_0^2 + a_1^2 + \frac{1}{5} - \frac{2}{3}a_0 + \frac{1}{2}a_1 \\
&= \frac{19}{720} \approx 0.026
\end{aligned}
$$

2 pts: piecewise simplify and expand
2 pts: primitive functions
1 pt: definite integral results
1 pt: plugging in a0 and a1.