25 March 2020 **Name:___________________**

ECE 438 **Exam No. 2** **Spring 2020**

- You have 24 hours to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is open book and open notes.
- Please do **NOT** discuss the problems with anyone else.

1. (25 pts) Consider the causal DT system described by the difference equation

$$y[n] = x[n] + \tfrac{1}{2}y[n-1]$$

a. (19) Use Z-transform techniques, including a partial fraction expansion, to find the response $y[n]$ of this system to the input $x[n] = u[-n] + \left(\frac{1}{2}\right)^n u[n]$, where $u[n]$ is the unit step function.

b. (6) Carefully sketch *by hand* the input $x[n]$ and the output $y[n]$, as functions of the independent variable n. Be sure to dimension both axes.

2. (25 pts) Fast Fourier Transform Algorithm

   a. (3) Calculate the *approximate* number of complex operations (COs) required to compute an 18-point DFT by directly evaluating the 18-point DFT sum. Here a complex operation is taken to mean 1 complex addition and 1 complex multiplication.

   b. (12) Derive a complete set of equations to show how an 18-point Discrete Fourier Transform (DFT) can be calculated in terms of two 3-point DFTs and one 2-point DFT via decimation-in-time.

   c. (5) Based on your answer to part b) above, list the complete ordering of the 18-point input to your 18-point FFT algorithm.

   d. (5) Based on your answer to part b) above, calculate the *approximate* number of complex operations (COs) required to compute an 18-point DFT using your FFT algorithm.

3. (25 pts) Spectral analysis via the DFT

Consider the 10-point signal $x[n] = \cos\left(\frac{7\pi}{10}n\right), n = 0,...,9$.

a. (15) Determine an exact expression for the 10-point discrete Fourier transform (DFT) $X^{(10)}\left[k\right], k = 0,...,9$ in terms of the function $\text{psinc}_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$.

b. (7) Based on your answer to part a) above, carefully sketch *by hand* the 10-point DFT $X^{(10)}\left[k\right], k = 0,...,9$. Be sure to dimension both axes. You may ignore the contribution of phase terms in your sketch.

c. (3) Discuss whether or not *picket fence effect* or *leakage* are present for this example, and explain how they manifest themselves in the 10-point DFT that you sketched in your answer to part b) above.

4. (25 pts) Consider a random variable X with density function

$$f_X(x) = \begin{cases} |x|, & 0 \le |x| \le 1 \\ 0, & \text{else} \end{cases} .$$

a. (3) Carefully sketch *by hand* the density function $f_X(x)$. Be sure to dimension both axes.

b. (4) Find the mean and variance of X.

c. (4) Suppose we generate a new random variable $Y = Q(X)$ by quantizing X according to the following 3-level *uniform* quantizer:

$$Q(x) = \begin{cases} -\frac{2}{3}, & -1 \le x < -\frac{1}{3} \\ 0, & -\frac{1}{3} \le x < \frac{1}{3} \\ \frac{2}{3}, & \frac{1}{3} \le x \le 1 \end{cases} .$$

Determine the *approximate* mean-squared quantization error $\varepsilon = E\left\{|Y - X|^2\right\}$ using the expression $\varepsilon_{\text{approx}} = \dfrac{\Delta^2}{12}$

d. (12) Now calculate the *exact* mean-squared quantization error $\varepsilon_{\text{exact}} = E\left\{|Y - X|^2\right\}$ for the 3-level *uniform* quantizer given above. Note that you can simplify your calculations by taking advantage of the symmetry of both $f_X(x)$ and $Q(x)$.

e. (2) Compare your answers to parts c) and d) above.

Problem 1

(25 pts) Consider the causal DT system described by the difference equation

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

a. (19) Use Z-transform techniques, including a partial fraction expansion, to find the response $y[n]$ of this system to the input $x[n] = u[-n] + (\frac{1}{2})^n u[n]$, where $u[n]$ is the unit step function.

b. (6) Carefully sketch by hand the input $x[n]$ and the output $y[n]$, as functions of the independent variable n. Be sure to dimension both axes.

a.

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$\text{Using } x[n-n_0] \stackrel{\text{ZT}}{\longleftrightarrow} z^{-n_0}X(z)$$

$$Y(z) = X(z) + \frac{1}{2}Y(z)z^{-1}$$

$$(1 - \frac{1}{2}z^{-1})Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \text{ since it is a causal system}$$

H(z): 1 pt: Z transfrom
1 pt: correct H(z) expression
1 pt: ROC of H(z)

Apply Z Transform to x[n], calculate Y(z), apply inverse Z Transform to Y(z)

Method 1:

$$x[n] = u[-n] + \left(\frac{1}{2}\right)^n u[n]$$

Using the following ZT pairs:

$$u[-n] \longleftrightarrow \frac{1}{1-z}, |z| < 1$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}, |z| > |a|$$

X(z): 3 pt: Z transform
1 pt: correct X(z) expression
1 pt: ROC of X(z)

we get

$$x[n] \overset{ZT}{\longleftrightarrow} X(z) = \frac{1}{1-z} + \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \frac{1}{2} < |z| < 1$$

$$\begin{aligned}
\Rightarrow Y(z) &= H(z)X(z) \\
&= \frac{1}{(1-\frac{1}{2}z^{-1})(1-z)} + \frac{1}{(1-\frac{1}{2}z^{-1})^2} \\
&= \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{z^{-1}})} + \frac{1}{(1-\frac{1}{2}z^{-1})^2} \\
&= \frac{-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} + \frac{1}{(1-\frac{1}{2}z^{-1})^2} \\
&= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}} + \frac{(2z)(\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})^2}, \qquad \frac{1}{2} < |z| < 1
\end{aligned}$$

Solve for A and B: $A - Az^{-1} + B - \frac{1}{2}Bz^{-1} = -z^{-1} \Rightarrow \begin{cases} A + B = 0 \\ -A - \frac{1}{2}B = -1 \end{cases}$

and get $A = 2$, $B = -2$.

$$\begin{aligned}
Y(z) &= \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-z^{-1}} + \frac{(2z)(\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})^2} \\
&= \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-z^{-1}} + (2z)\frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \qquad \frac{1}{2} < |z| < 1
\end{aligned}$$

Using the following ZT pairs:

$$\begin{aligned}
a^n u[n] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| > |a| \\
-a^n u[-n-1] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| < |a| \\
na^n u[n] &\longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, |z| > |a| \\
x[n-n_0] &\longleftrightarrow z^{-n_0}X(z)
\end{aligned}$$

y[n]: 1 pt: Y(z) = H(z)X(z)
2 pt: expansion of Y(z)
2 pt: solving for A and B
1 pt: ROC of Y(z)

4 pt: Z transform
1 pt: correct y[n] expression

we get

$$\begin{aligned}
Y(z) \overset{ZT}{\longleftrightarrow} y[n] &= 2\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1] + 2(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1] \\
&= 2\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1] + (n+1)\left(\frac{1}{2}\right)^n u[n+1], \qquad \frac{1}{2} < |z| < 1
\end{aligned}$$

Method 2:

$$\begin{aligned} x[n] &= u[-n] + \left(\frac{1}{2}\right)^n u[n] \\ &= \delta[n] + u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \end{aligned}$$

Using the following ZT pairs:

$$\begin{aligned} \delta[n] &\longleftrightarrow 1, \text{all z} \\ a^n u[n] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| > |a| \\ -a^n u[-n-1] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| < |a| \end{aligned}$$

we get

$$\text{x[n]} \stackrel{\text{ZT}}{\longleftrightarrow} X(z) = 1 - \tfrac{1}{1-z^{-1}} + \tfrac{1}{1-\frac{1}{2}z^{-1}}, \qquad \tfrac{1}{2} < |z| < 1$$

$$\begin{aligned} \Rightarrow Y(z) &= H(z)X(z) \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} + \frac{1}{(1-\frac{1}{2}z^{-1})^2} \\ &= \frac{2}{1-\frac{1}{2}z^{-1}} - \left(\frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}}\right) + \left(\frac{1}{(1-\frac{1}{2}z^{-1})^2} - \frac{1}{1-\frac{1}{2}z^{-1}}\right) \\ &= \frac{2}{1-\frac{1}{2}z^{-1}} - \left(\frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}}\right) \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \qquad \frac{1}{2} < |z| < 1 \end{aligned}$$

Solve for A and B: $A - Az^{-1} + B - \frac{1}{2}Bz^{-1} = 1 \Rightarrow \begin{cases} A + B = 1 \\ -A - \frac{1}{2}B = 0 \end{cases}$

and get $A = -1$, $B = 2$.

$$\begin{aligned} Y(z) &= \frac{2}{1-\frac{1}{2}z^{-1}} - \left(\frac{-1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-z^{-1}}\right) + \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \\ &= \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-z^{-1}} + \frac{\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})^2}, \qquad \frac{1}{2} < |z| < 1 \end{aligned}$$

Using the following ZT pairs:

$$
\begin{aligned}
a^n u[n] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| > |a| \\
-a^n u[-n-1] &\longleftrightarrow \frac{1}{1-az^{-1}}, |z| < |a| \\
na^n u[n] &\longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, |z| > |a|
\end{aligned}
$$

we get

$$
\begin{aligned}
Y(z) \xleftrightarrow{\text{ZT}} y[n] &= 3\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1] + n\left(\frac{1}{2}\right)^n u[n] \\
&= (3+n)\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1]
\end{aligned}
$$

b. Sketch:

$$x[n] = u[-n] + \left(\frac{1}{2}\right)^n u[n]$$

$$y[n] = (3+n)\left(\frac{1}{2}\right)^n u[n] + 2u[-n-1]$$

x[n]:

$n < 0$: $x[n] = 1$

$x[0] = 1 + 1 = 2$ $\quad$ $x[1] = \frac{1}{2}$ $\quad$ $x[2] = (\frac{1}{2})^2 = \frac{1}{4}$

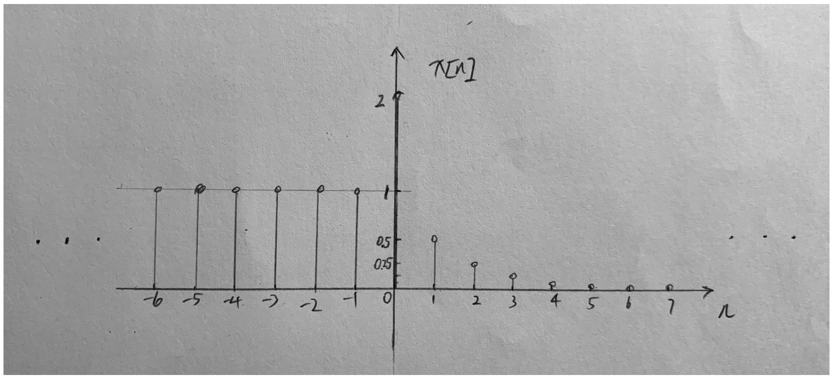
$x[3] = (\frac{1}{2})^3 = \frac{1}{8}$ $\quad$ $x[4] = (\frac{1}{2})^4 = \frac{1}{16}$ $\quad$ ......

y[n]:

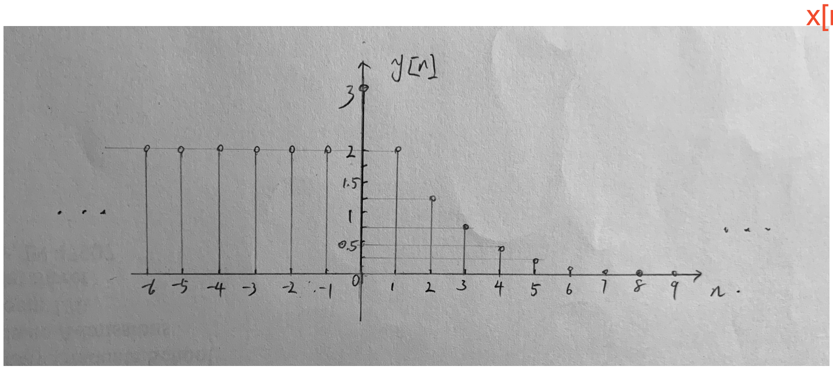
$n < 0$: $y[n] = 2$

$y[0] = 3 + 0 = 3$ $\quad$ $y[1] = (3+1)\frac{1}{2} = 2$ $\quad$ $y[2] = (3+2)(\frac{1}{2})^2 = \frac{5}{4}$

$y[3] = (3+3)(\frac{1}{2})^3 = \frac{3}{4}$ $\quad$ $y[4] = (3+4)(\frac{1}{2})^4 = \frac{7}{16}$ $\quad$ ......



x[n]: 2 pt: correct values
1 pt: dimension both axes



y[n]: 2 pt: correct values
1 pt: dimension both axes

Problem 2

(25 points) Fast Fourier Transform Algorithm

(a) **(3 points)** Calculate the *approximate* number of complex operations (COs) required to compute an 18-point DFT by directly evaluating the 18-point DFT sum. Here a complex operation is taken to mean 1 complex addition and 1 complex multiplication.

(b) **(12 points)** Derive a complete set of equations to show how an 18-point Discrete Fourier Transform (DFT) can be calculated in terms of three stages: two stages of 3-point DFTs and one stage of 2-point DFT via decimation-in-time.[1]

(c) **(5 points)** Based on your answer to part b) above, list the complete ordering of the 18-point input to your 18-point FFT algorithm.

(d) **(5 points)** Based on your answer to part b) above, calculate the *approximate* number of complex operations (COs) required to compute an 18-point DFT using your FFT algorithm.

(a) The DFT formula is

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

To calculate each X_k, N multiplications and $N-1$ additions are required, so there are approximately N COs when N is large.

Therefore, there are approximately $N \times N = N^2 = 324$ COs.

Correct number of COs: 3 points

(b) Again, the DFT formula is

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{17} x[n] e^{-j\frac{2\pi kn}{18}} \end{aligned}$$

[1] This part has been restated by Professor Allebach in Piazza.

(a) 4 points **Stage III**

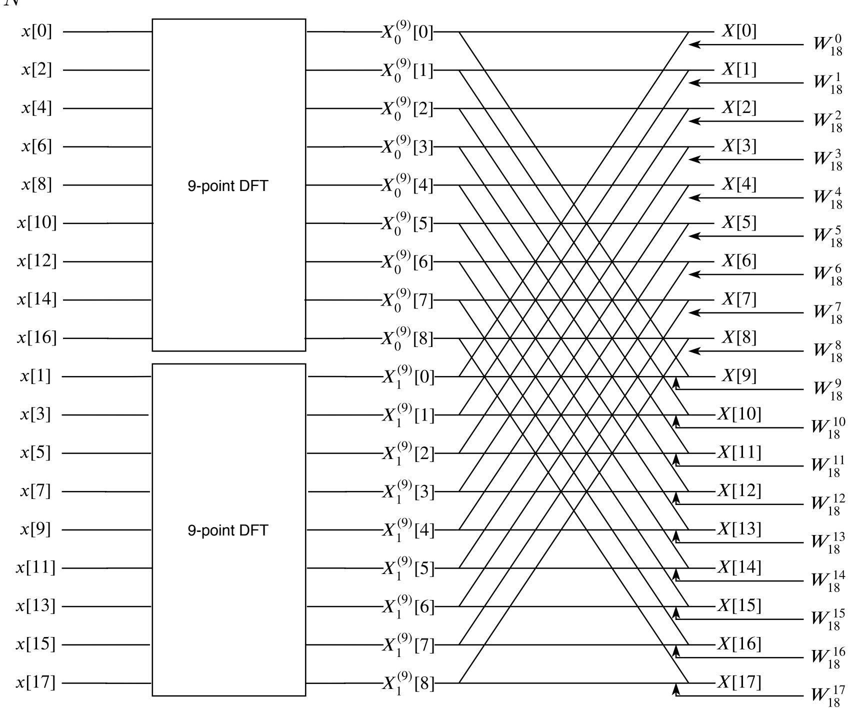
Let $n = 2l + p$, where $l = 0, 1, \ldots, 8$ and $p = 0, 1$,

$$\begin{aligned}
X^{(18)}[k] &= \sum_{n=0}^{17} x[n] e^{-j\frac{2\pi kn}{18}} \\
&= \sum_{l=0}^{8} x[2l] e^{-j\frac{2\pi k 2l}{18}} + \sum_{l=0}^{8} x[2l+1] e^{-j\frac{2\pi k(2l+1)}{18}} \\
&= \sum_{l=0}^{8} x[2l] e^{-j\frac{2\pi kl}{9}} + e^{-j\frac{2\pi k}{18}} \sum_{l=0}^{8} x[2l+1] e^{-j\frac{2\pi kl}{9}} \\
&= \sum_{l=0}^{8} x[2l] e^{-j\frac{2\pi kl}{9}} + W_{18}^{k} \sum_{l=0}^{8} x[2l+1] e^{-j\frac{2\pi kl}{9}}
\end{aligned}$$

Let $x_p[l] = x[2l + p]$,

$$\begin{aligned}
X^{(18)}[k] &= \sum_{l=0}^{8} x[2l] e^{-j\frac{2\pi kl}{9}} + W_{18}^{k} \sum_{l=0}^{8} x[2l+1] e^{-j\frac{2\pi kl}{9}} \\
&= \sum_{l=0}^{8} x_0[l] e^{-j\frac{2\pi kl}{9}} + W_{18}^{k} \sum_{l=0}^{8} x_1[l] e^{-j\frac{2\pi kl}{9}} \\
&= X_0^{(9)}[k] + W_{18}^{k} X_1^{(9)}[k],
\end{aligned}$$

where $W_N^k = e^{-j\frac{2\pi k}{N}}$.



The flow diagram is not required.

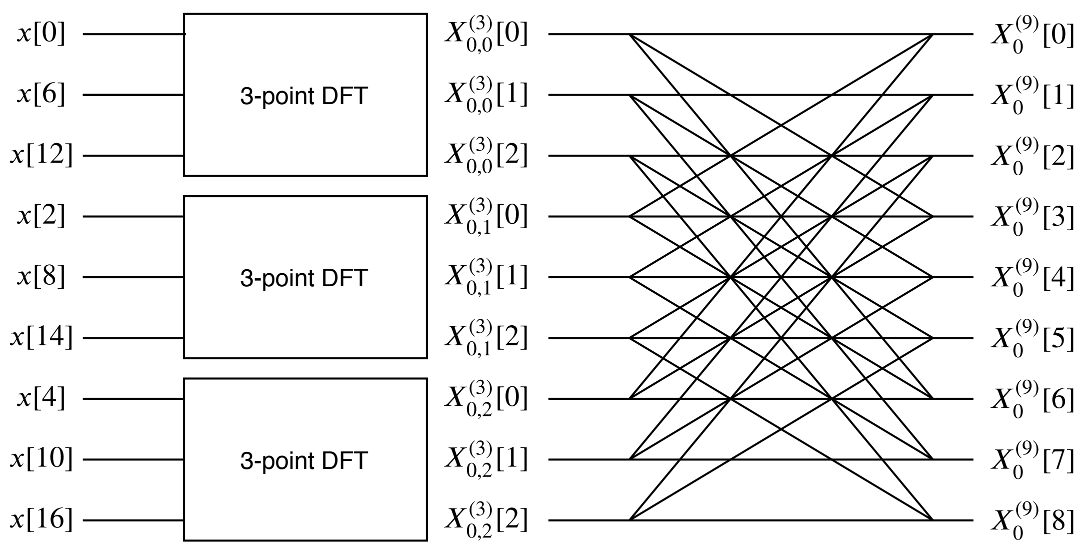
(b) 4 points **Stage II**

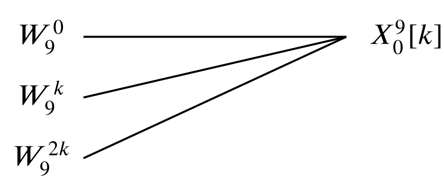
Let $l = 3m + q$, where $m = 0, 1, 2$ and $q = 0, 1, 2$.

$$
\begin{aligned}
X_p^{(9)}[k] &= \sum_{l=0}^{9} x_p[l] e^{-j\frac{2\pi k l}{9}} \\
&= \sum_{m=0}^{2} x_p[3m] e^{-j\frac{2\pi k 3m}{9}} + \sum_{m=0}^{2} x_p[3m+1] e^{-j\frac{2\pi k(3m+1)}{9}} + \sum_{m=0}^{2} x_p[3m+2] e^{-j\frac{2\pi k(3m+2)}{9}} \\
&= \sum_{m=0}^{2} x_p[3m] e^{-j\frac{2\pi k m}{3}} + e^{-j\frac{2\pi k}{9}} \sum_{m=0}^{2} x_p[3m+1] e^{-j\frac{2\pi k m}{3}} + e^{-j\frac{2\pi 2k}{9}} \sum_{m=0}^{2} x_p[3m+2] e^{-j\frac{2\pi k m}{3}} \\
&= \sum_{m=0}^{2} x_p[3m] e^{-j\frac{2\pi k m}{3}} + W_9^k \sum_{m=0}^{2} x_p[3m+1] e^{-j\frac{2\pi k m}{3}} + W_9^{2k} \sum_{m=0}^{2} x_p[3m+2] e^{-j\frac{2\pi k m}{3}}
\end{aligned}
$$

Let $x_{p,q}[m] = x_p[3m + q]$,

$$
\begin{aligned}
X_p^{(9)}[k] &= \sum_{m=0}^{2} x_{p,0}[m] e^{-j\frac{2\pi k m}{3}} + W_9^k \sum_{m=0}^{2} x_{p,1}[m] e^{-j\frac{2\pi k m}{3}} + W_9^{2k} \sum_{m=0}^{2} x_{p,2}[m] e^{-j\frac{2\pi k m}{3}} \\
&= X_{p,0}^{(3)}[k] + W_9^k X_{p,1}^{(3)}[k] + W_9^{2k} X_{p,2}^{(3)}[k]
\end{aligned}
$$

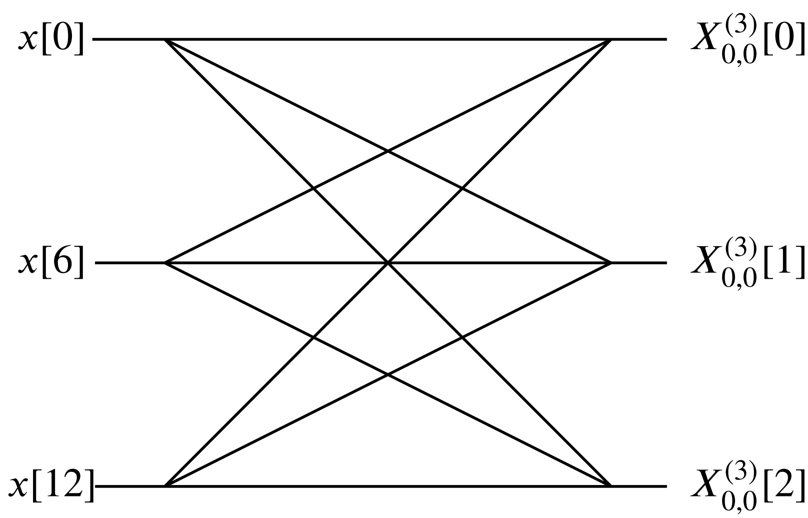


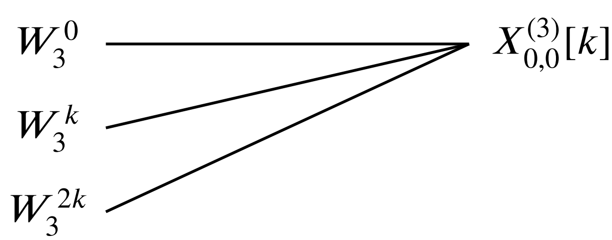


The flow diagram is not required.

(c) 4 points **Stage I**

$$\begin{aligned} X_{p,q}^{(3)}[k] &= \sum_{m=0}^{2} x_{p,q}[m] e^{-j\frac{2\pi k m}{3}} \\ &= x_{p,q}[0] + x_{p,q}[1] e^{-j\frac{2\pi k}{3}} + x_{p,q}[2] e^{-j\frac{2\pi 2k}{3}} \\ &= x_{p,q}[0] + W_3^k x_{p,q}[1] + W_3^{2k} x_{p,q}[2] \end{aligned}$$





The flow diagram is not required.

(c) Since

$$x_p[l] = x[2l + p] \quad (1)$$
$$x_{p,q}[m] = x_p[3m + q] \quad (2)$$

substitute (1) into (2), we get

$$x_{p,q}[m] = x[6m + 2q + p],$$

where $m = 0, 1, 2, q = 0, 1, 2, p = 0, 1$.

p	q	m	$x_{p,q}[m]$	$6m + 2q + p$	$x[6m + 2q + p]$
0	0	0	$x_{0,0}[0]$	0	$x[0]$
0	0	1	$x_{0,0}[1]$	6	$x[6]$
0	0	2	$x_{0,0}[2]$	12	$x[12]$
0	1	0	$x_{0,1}[0]$	2	$x[2]$
0	1	1	$x_{0,1}[1]$	8	$x[8]$
0	1	2	$x_{0,1}[2]$	14	$x[14]$
0	2	0	$x_{0,2}[0]$	4	$x[4]$
0	2	1	$x_{0,2}[1]$	10	$x[10]$
0	2	2	$x_{0,2}[2]$	16	$x[16]$
1	0	0	$x_{1,0}[0]$	1	$x[1]$
1	0	1	$x_{1,0}[1]$	7	$x[7]$
1	0	2	$x_{1,0}[2]$	13	$x[13]$
1	1	0	$x_{1,1}[0]$	3	$x[3]$
1	1	1	$x_{1,1}[1]$	9	$x[9]$
1	1	2	$x_{1,1}[2]$	15	$x[15]$
1	2	0	$x_{1,2}[0]$	5	$x[5]$
1	2	1	$x_{1,2}[1]$	11	$x[11]$
1	2	2	$x_{1,2}[2]$	17	$x[17]$

Table 1: Input Ordering is shown in the 6^{th} column from top to bottom

Correct ordering: 5 points

(d) In the Stage 3, there are 18 multiplications and 18 additions. So there are 18 COs.

Correct number of COs in Stage 3: 2 points

In the Stage 2, there are 2 multiplications and 2 additions for each of the point in a 9-point DFT, so there are $2 \times 9 \times 2 = 36$ COs.

Correct number of COs in Stage 2: 2 points

In the Stage 1, there are 2 multiplications and 2 additions for each of the point in a 3-point DFT, so there are $2 \times 3 \times 6 = 36$ COs.

Correct number of COs in Stage 1: 1 points

Totally, there are approximately $18 + 36 + 36 = 90$ COs.

3. (25 pts) Spectral analysis via the DFT

Consider the 10-point signal $x[n] = \cos\left(\frac{7\pi}{10}n\right), n = 0, \dots, 9$.

a. (15) Determine an exact expression for the 10-point discrete Fourier transform (DFT) $X^{(10)}[k], k = 0, \dots, 9$ in terms of the function $psinc_N(\omega) = \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}$.

$$\cos\left(\frac{7\pi}{10}n\right) = \frac{1}{2}\left(e^{j\frac{7\pi}{10}n} + e^{-j\frac{7\pi}{10}n}\right)$$

$$w[n] = u[n] - u[n-10] \overset{DTFT}{\Longleftrightarrow} psinc_{10}(\omega)e^{-j\omega\frac{9}{2}} = \frac{\sin(\frac{10\omega}{2})}{\sin(\frac{\omega}{2})}e^{-j\omega\frac{9}{2}}$$

$$x[n] = w[n] \times \frac{1}{2}\left(e^{j\frac{7\pi}{10}n} + e^{-j\frac{7\pi}{10}n}\right) = \frac{1}{2}\left(w[n]e^{j\frac{7\pi}{10}n} + w[n]e^{-j\frac{7\pi}{10}n}\right)$$

By the modulation property of the DTFT, we get:

$$X(\omega) = \frac{1}{2}\left(psinc_{10}\left(\omega - \frac{7\pi}{10}\right)e^{-j(\omega - \frac{7\pi}{10})\frac{9}{2}} + psinc_{10}\left(\omega + \frac{7\pi}{10}\right)e^{-j(\omega + \frac{7\pi}{10})\frac{9}{2}}\right)$$

We get the 10-point DFT when we take samples of the DTFT where $\omega = \frac{2\pi}{10}k, k = 0, \dots, 9$.

$$X^{(10)}[k] = \frac{1}{2}\left(psinc_{10}\left(\frac{2\pi}{10}k - \frac{7\pi}{10}\right)e^{-j(\frac{2\pi}{10}k - \frac{7\pi}{10})\frac{9}{2}} + psinc_{10}\left(\frac{2\pi}{10}k + \frac{7\pi}{10}\right)e^{-j(\frac{2\pi}{10}k + \frac{7\pi}{10})\frac{9}{2}}\right), k = 0, \dots, 9$$

Since $\frac{7\pi}{10} = \frac{2\pi(3.5)}{10}$, we can rewrite this as:

$$X^{(10)}[k] = \frac{1}{2}\left(psinc_{10}\left(\frac{2\pi}{10}(k-3.5)\right)e^{-j\frac{2\pi}{10}(k-3.5)\frac{9}{2}} + psinc_{10}\left(\frac{2\pi}{10}(k+3.5)\right)e^{-j\frac{2\pi}{10}(k+3.5)\frac{9}{2}}\right),$$

$$k = 0, \dots, 9$$

We normally don't like seeing $k + k_0$ in a DFT, and since $psinc_{10}(\cdot)$ is periodic in relation to k with period 10, we can rewrite it as:

$$psinc_{10}\left(\frac{2\pi}{10}(k+3.5)\right) = psinc_{10}\left(\frac{2\pi}{10}(k-10+3.5)\right) = psinc_{10}\left(\frac{2\pi}{10}(k-6.5)\right)$$

So,

$$X^{(10)}[k] = \frac{1}{2}\left(psinc_{10}\left(\frac{2\pi}{10}(k-3.5)\right)e^{-j\frac{2\pi}{10}(k-3.5)\frac{9}{2}} + psinc_{10}\left(\frac{2\pi}{10}(k-6.5)\right)e^{-j\frac{2\pi}{10}(k-6.5)\frac{9}{2}}\right),$$

$$k = 0, \dots, 9$$

3 pts. – Correct DTFT of w[n]

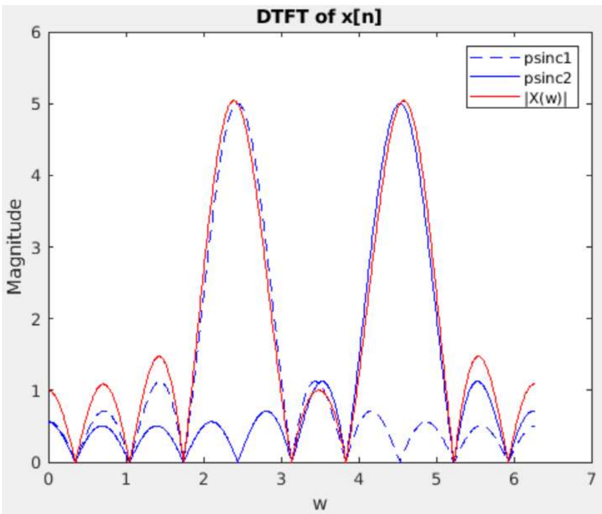
3 pts. – Correct DTFT of x[n]

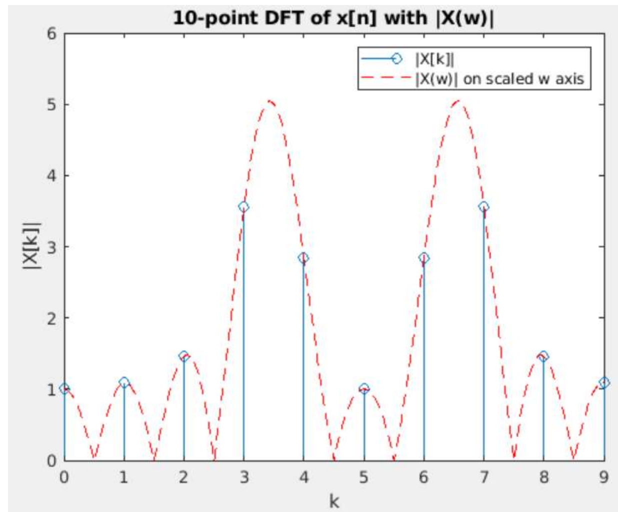
4 pts. – Correct DFT frequency shift k_0 (2 pts. each)

5 pts. – Correct expression of DFT of x[n]

b. (7) Based on your answer to part a) above, carefully sketch *by hand* the 10-point DFT $X^{(10)}[k], k = 0, \ldots, 9$. Be sure to dimension both axes. You may ignore the contribution of phase terms in your sketch.

Note: This solution keeps the contribution of phase terms.





4 pts. Correct DFT sketch

2 pts. Dimension axes (1 pt. each)

1 pt. Label axes

c. (3) Discuss whether or not *picket fence effect* or *leakage* are present for this example, and explain how they manifest themselves in the 10-point DFT that you sketched in your answer to part b) above.

Leakage is caused by the truncation of a signal, since the signal $\cos\left(\frac{7\pi}{10}n\right)$ is truncated to only be on the interval $n = 0, \ldots, 9$, this example does experience leakage. The leakage is manifest in the DFT by the $psinc_{10}(\cdot)$.

The picket fence effect is caused by sampling in the frequency domain. For a sinusoidal signal, it appears if the frequency of the sinusoid, ω_0, is not an integer multiple of $\frac{2\pi}{N}$. In this case, $N = 10$, and the frequency of the sinusoid is $\omega_0 = \frac{7\pi}{10} = \frac{2\pi(3.5)}{10}$. The picket fence effect is manifest in this DFT by the samples being offset from the center of the main lobe of each $psinc_{10}(\cdot)$.

1 pt. – Saying leakage is present

1 pt. – Saying picket fence effect is present

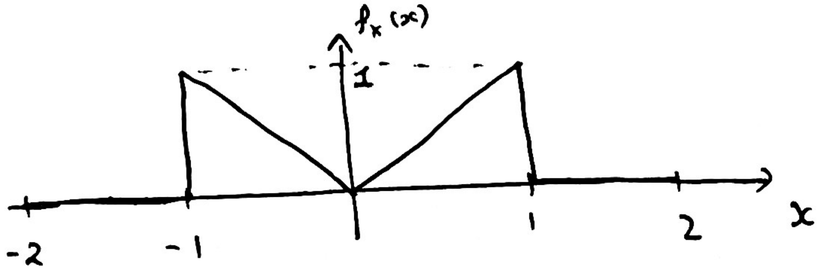
0.5 pt. – Explain how leakage is manifest

0.5 pt. – Explain how picket fence effect is manifest

4)

$$f_X(x) = \begin{cases} |x| & 0 \le |x| \le 1 \\ 0 & \text{else} \end{cases}$$

a.



b. $E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_{-1}^{1} x\,|x|\,dx = \int_{-1}^{0} x\cdot(-x)\,dx + \int_{0}^{1} x\cdot x\,dx$

$= \int_{-1}^{0} -x^2\,dx + \int_{0}^{1} x^2\,dx = -\left(0 - \frac{-1}{3}\right) + \left(\frac{1}{3} - 0\right) = 0$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-1}^{0} x^2\cdot(-x)\,dx + \int_{0}^{1} x^2\cdot x\,dx$

$= -\frac{x^4}{4}\Big]_{-1}^{0} + \frac{x^4}{4}\Big]_{0}^{1} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\mu_X = E[X] = 0 \qquad \sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{2}$

c. $$Q(x) = \begin{cases} -2/3 & -1 \le x \le -\frac{1}{3} \\ 0 & -\frac{1}{3} \le x \le 1/3 \\ 2/3 & 1/3 \le x \le 1 \end{cases}$$

$\Delta = \frac{2}{3} \qquad \varepsilon_{approx} = \frac{\Delta^2}{12} = \frac{4}{9\cdot 12} = \frac{1}{27}$

d. $\mathcal{E}_{exact} = E[|Y-X|^2] = \int_{-\infty}^{\infty} (Q(x) - x)^2 f_X(x)\,dx$

$$= \int_{-1}^{-\frac{1}{3}} \left(-\tfrac{2}{3} - x\right)^2 |x|\,dx + \int_{-\frac{1}{3}}^{1/3} (0 - x)^2 |x|\,dx + \int_{1/3}^{1} \left(\tfrac{2}{3} - x\right)^2 |x|\,dx$$

$$\int_{-1}^{-\frac{1}{3}} \left(\tfrac{2}{3} + x\right)^2 (-x)\,dx = \int_{-1}^{-\frac{1}{3}} -\tfrac{4}{9}x - \tfrac{4}{3}x^2 - x^3\,dx = \left(-\tfrac{2}{9}x^2 - \tfrac{4}{9}x^3 - \tfrac{x^4}{4}\right)\Big]_{-1}^{-1/3}$$

$$= -\tfrac{2}{9}\cdot\tfrac{1}{9} + \tfrac{4}{9}\cdot\tfrac{1}{27} - \tfrac{1}{4}\cdot\tfrac{1}{81} + \tfrac{2}{9} - \tfrac{4}{9} + \tfrac{1}{4} = \tfrac{4}{243}$$

$$\int_{1/3}^{1} \left(\tfrac{2}{3} - x\right)^2 x\,dx = \int_{1/3}^{1} \tfrac{4}{9}x - \tfrac{4}{3}x^2 + x^3\,dx = \left(\tfrac{2}{9}x^2 - \tfrac{4}{9}x^3 + \tfrac{x^4}{4}\right)\Big]_{1/3}^{1}$$

$$= \tfrac{2}{9} - \tfrac{4}{9} + \tfrac{1}{4} - \tfrac{2}{9}\cdot\tfrac{1}{9} + \tfrac{4}{9}\cdot\tfrac{1}{27} - \tfrac{1}{4}\cdot\tfrac{1}{81} = \tfrac{4}{243}$$

$$\int_{-\frac{1}{3}}^{1/3} x^2 |x|\,dx = \int_{-\frac{1}{3}}^{0} -x^3\,dx + \int_{0}^{1/3} x^3\,dx = -\tfrac{x^4}{4}\Big]_{-\frac{1}{3}}^{0} + \tfrac{x^4}{4}\Big]_{0}^{1/3}$$

$$= \tfrac{1}{324} + \tfrac{1}{324} = \tfrac{1}{162}$$

$$\mathcal{E}_{exact} = \tfrac{4}{243} + \tfrac{4}{243} + \tfrac{1}{162} = \tfrac{19}{486}$$

e. $\mathcal{E}_{exact} \approx \frac{19}{486}$

$\mathcal{E}_{approx} = \frac{1}{27} = \frac{18}{486}$

Approximate and exact quantization errors are very close.