19 February 2020 **Name:____________________**

ECE 438 **Exam No. 1** **Spring 2020**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes. Smart watches and mobile phones must be put away.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n]=x[n]+2x[n-1]+x[n-2].$$

a. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

b. (5) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

c. (5) Find a simple expression for the phase $\angle H(\omega)$ of the frequency response.

d. (5) Carefully sketch $|H(\omega)|$ and $\angle H(\omega)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

2. (25 pts.) Consider a linear, time-invariant system with unit sample (impulse) response

$$h[n]=\begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{else} \end{cases} .$$

a. (20) Find the response of this system $y[n]$ to the input

$$x[n]=\left(\frac{1}{2}\right)^n u[n],$$

by evaluating the convolution $y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

Your solution for $y[n]$ should be an analytical expression or expression(s) for the signal. It should not contain any summation signs $\sum$.

b. (5) Carefully sketch the function $y[n]$.

3. (25) Consider the real-valued continuous-time signal $x(t)$ defined by

$$x(t) = \begin{cases} \sin(2\pi t/2), & |t| < 1 \\ 0, & \text{else} \end{cases}$$

a. (5) Carefully sketch $x(t)$ being sure to dimension both axes.

b. (15) Find a simple expression for the CTFT $X(f)$ of $x(t)$. Your answer should not include any operators, such as convolution, rep, or comb.

c. (5) Carefully sketch $X(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes

4. (25 pts) Consider the real-valued continuous-time signal $x(t) = \cos\left(2\pi(6000)t\right)$.

This signal is sampled at an 8 kHz rate with an ideal sampler (no prefilter; so it is not an ideal A/D) to generate the continuous-time (CT) sampled signal

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(nT)\delta(t - nT),$$

where T is the sampling interval.

a. (10) Find a simple expression for the CTFT $X_s(f)$ of $x_s(t)$. Your final answer should not contain any operators, such as convolution, rep, or comb.

b. (5) Carefully sketch $X_s(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

Suppose we reconstruct a continuous-time signal $y(t)$ from $x_s(t)$ by filtering it with an ideal low-pass filter with cutoff frequency f_c = 4 kHz, and gain T in the passband.

c. (10) Find a simple expression for $y(t)$. Your final answer should not contain any operators, such as convolution, rep, or comb.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

a. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

b. (5) Find a simple expression for the magnitude $|H(\omega)|$ of the frequency response.

c. (5) Find a simple expression for the phase $\underline{/H(\omega)}$ of the frequency response.

d. (5) Carefully sketch $|H(\omega)|$ and $\underline{/H(\omega)}$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

(a) There are two main methods of computing *H(ω)*:

Method 1:

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

$\Updownarrow$ DTFT

$$Y(\omega) = X(\omega) + 2e^{-j\omega}X(\omega) + e^{-j2\omega}X(\omega)$$

$$Y(\omega) = X(\omega)\left[1 + 2e^{-j\omega} + e^{-j2\omega}\right]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

We can further simplify $H(\omega)$ by using Euler's formula:

$$H(\omega) = e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$H(\omega) = e^{-j\omega}(2 + 2\cos\omega)$$

$$H(\omega) = 2e^{-j\omega}(1 + \cos\omega)$$

3 pts: Correct expression of *Y(ω)*

5 pts: Correct expression of *H(ω)* before simplification

2 pts: Correct expression of *H(ω)* after simplification

Method 2:

Since $h[n]$ is defined as the output when $x[n] = \delta[n]$,

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$\Updownarrow \text{DTFT}$$

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

We can further simplify $H(\omega)$ by using Euler's formula:

$$H(\omega) = e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega})$$

$$H(\omega) = e^{-j\omega}(2 + 2\cos\omega)$$

$$H(\omega) = 2e^{-j\omega}(1 + \cos\omega)$$

3 pts: Correct expression of h[n]

5 pts: Correct expression of *H(ω)* before simplification

2 pts: Correct simplified expression of *H(ω)*

(b)

$$|H(\omega)| = |2e^{-j\omega}(1 + \cos\omega)|$$

$$|H(\omega)| = |2||e^{-j\omega}||1 + \cos\omega|$$

Since $|e^{-j\omega}| = 1$ and $1 + \cos\omega$ is always positive,

$$|H(\omega)| = 2 + 2\cos\omega$$

1 pt: Magnitude of a product equals product of magnitudes

1 pt: Correct evaluation of magnitude of complex exponential

1 pt: Determining $1 + \cos\omega \geq 0$ for all ω

2 pts: Correct final answer

(c)

$$\angle H(w) = \angle[2e^{-jw}(1+\cos w)]$$

When complex numbers are multiplied, their phases add.

$$\angle H(w) = \angle 2 + \angle e^{-jw} + \angle(1+\cos w)$$

phase is computed as $\angle(a+jb) = \tan^{-1}\left(\frac{b}{a}\right) = \theta$

$$\angle H(w) = \tan^{-1}\left(\frac{0}{2}\right) - w + \tan^{-1}\left(\frac{0}{1+\cos w}\right)$$

$$\angle H(w) = \tan^{-1}(0) - w + \tan^{-1}(0)$$

$$\angle H(w) = 0 - w + 0$$

$$\angle H(w) = -w$$

The function tan^{-1}(b/a) can only return an angle between -pi and pi. Since 1 +cos(w) is always non-negative, this works here. But if a < 0 and b = 0, then the correct phase is pi or -pi radians, not 0 radians. The C language handles this by providing the function atan2(a,b).

1 pt: Phase of a product equals sum of phases

1 pt: Correct evaluation of phase of complex exponential

1 pt: Phase of a positive, real value is 0

2 pts: Correct final answer

(d)

To sketch $|H(\omega)|$, let's compute some of it's values
Since $\cos(-\omega) = \cos\omega$, we can compute just $\omega \geq 0$.

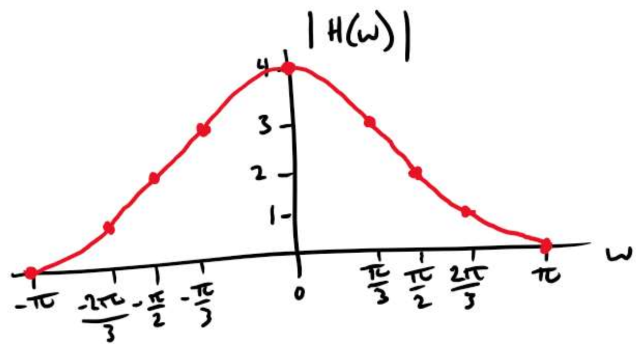
$\omega = 0 \qquad |H(0)| = 2 + 2\cos(0) = 2 + 2(1) = 4$

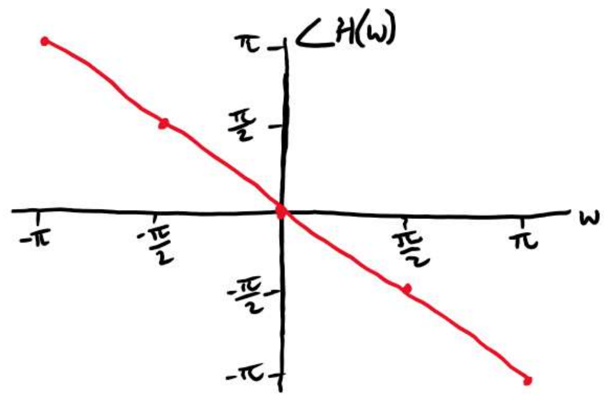
$\omega = \frac{\pi}{3} \qquad |H(\frac{\pi}{3})| = 2 + 2\cos(\frac{\pi}{3}) = 2 + 2(\frac{1}{2}) = 3$

$\omega = \frac{\pi}{2} \qquad |H(\frac{\pi}{2})| = 2 + 2\cos(\frac{\pi}{2}) = 2 + 2(0) = 2$

$\omega = \frac{2\pi}{3} \qquad |H(\frac{2\pi}{3})| = 2 + 2\cos(\frac{2\pi}{3}) = 2 + 2(-\frac{1}{2}) = 1$

$\omega = \pi \qquad |H(\pi)| = 2 + 2\cos(\pi) = 2 + 2(-1) = 0$





1 pt: Plot of $|H(\omega)|$

1 pt: Plot of $\angle H(\omega)$

1 pt: Important quantities (red dots)

1 pt: y-axis label

1 pt: x-axis label

2. Consider a linear, time-invariant system with unit sample (impulse) response

$$h[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{else} \end{cases}$$

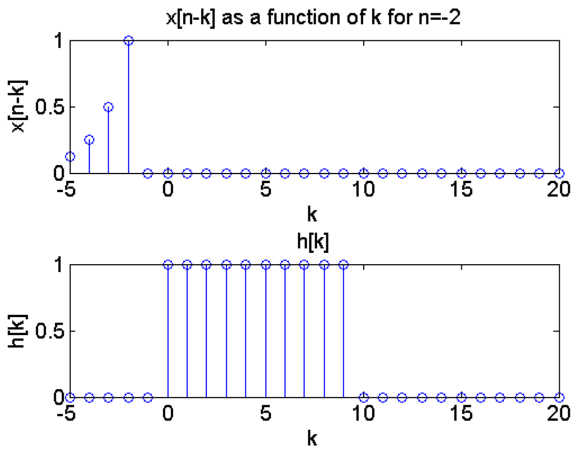
a. Find the response of this system $y[n]$ to the input

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=0}^{9} x[n-k] \end{aligned}$$

2 pts

Case 1: $n < 0$:

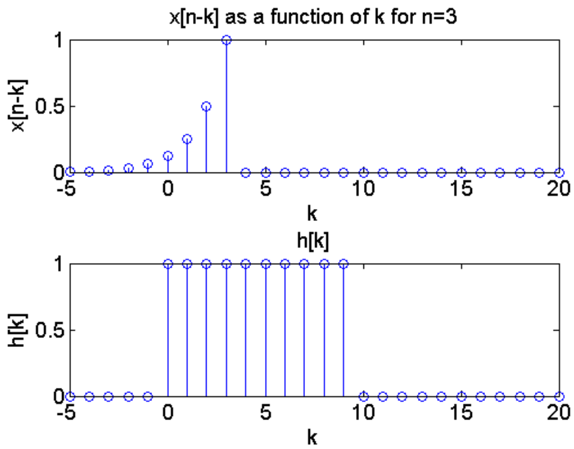


2 pts

$y[n] = 0$

3 pts

Case 2: $0 \leq n < 9$:



3 pts

$$y[n] = \sum_{k=0}^{9} \left(\frac{1}{2}\right)^{n-k} u[n-k] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k}$$

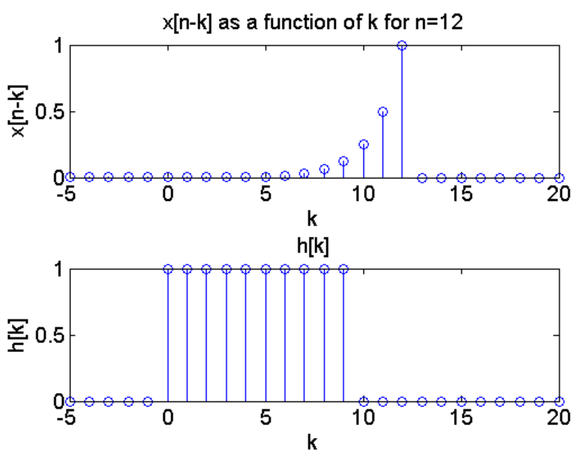
$$= \sum_{k=0}^{n} (2)^{k-n} = 2^{-n} \frac{1-2^{n+1}}{1-2}$$

$$= 2 - 2^{-n} = 2 - \left(\frac{1}{2}\right)^{n}$$

2 pts

3 pts

Case 3: $n \geq 9$:



$$y[n] = \sum_{k=0}^{9} \left(\frac{1}{2}\right)^{n-k} u[n-k] = \sum_{k=0}^{9} \left(\frac{1}{2}\right)^{n-k}$$

3 pts

$$= \sum_{k=0}^{9} (2)^{k-n} = 2^{-n} \frac{1-2^{9+1}}{1-2}$$

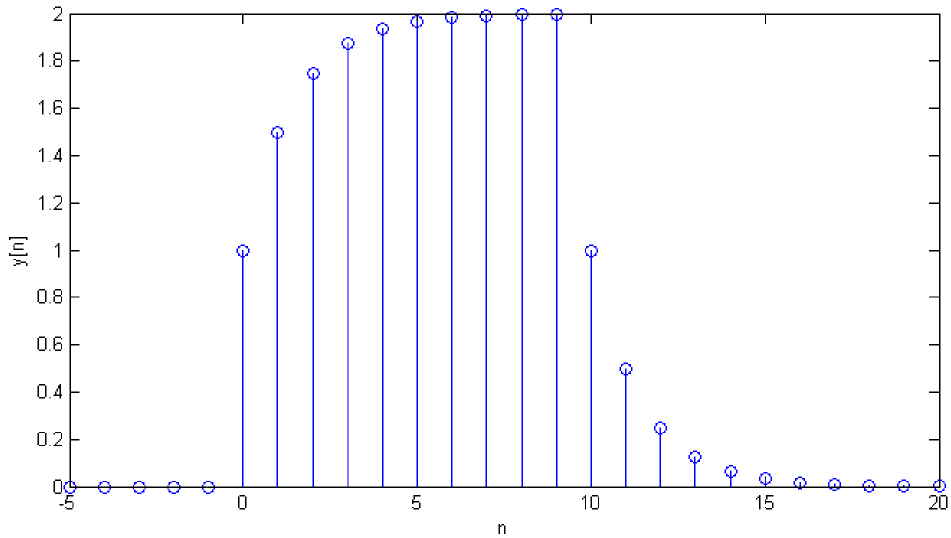
$$= 2^{-n}(2^{10}-1) = 1023 \times 2^{-n}$$

$$= 2^{10-n} - 2^{-n} = \left(\frac{1}{2}\right)^{n-10} - \left(\frac{1}{2}\right)^{n}$$

2 pts

b. Carefully sketch the function $y[n]$.

5 pts



Problem 3

Consider the real-valued continuous-time signal $x(t)$ defined by

$$x(t) = \begin{cases} \sin(2\pi t/2) & |t| < 1 \\ 0 & \text{else} \end{cases}$$

(a) **(5 pts)** Carfully sketch $x(t)$ being sure to dimension both axes.

(b) **(15 pts)** Find a simple expression for the CTFT $X(f)$ of $x(t)$. Your answer should not include any operators, such as convolution, rep or comb.

(c) **(5 pts)** Carefully sketch $X(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

(a)

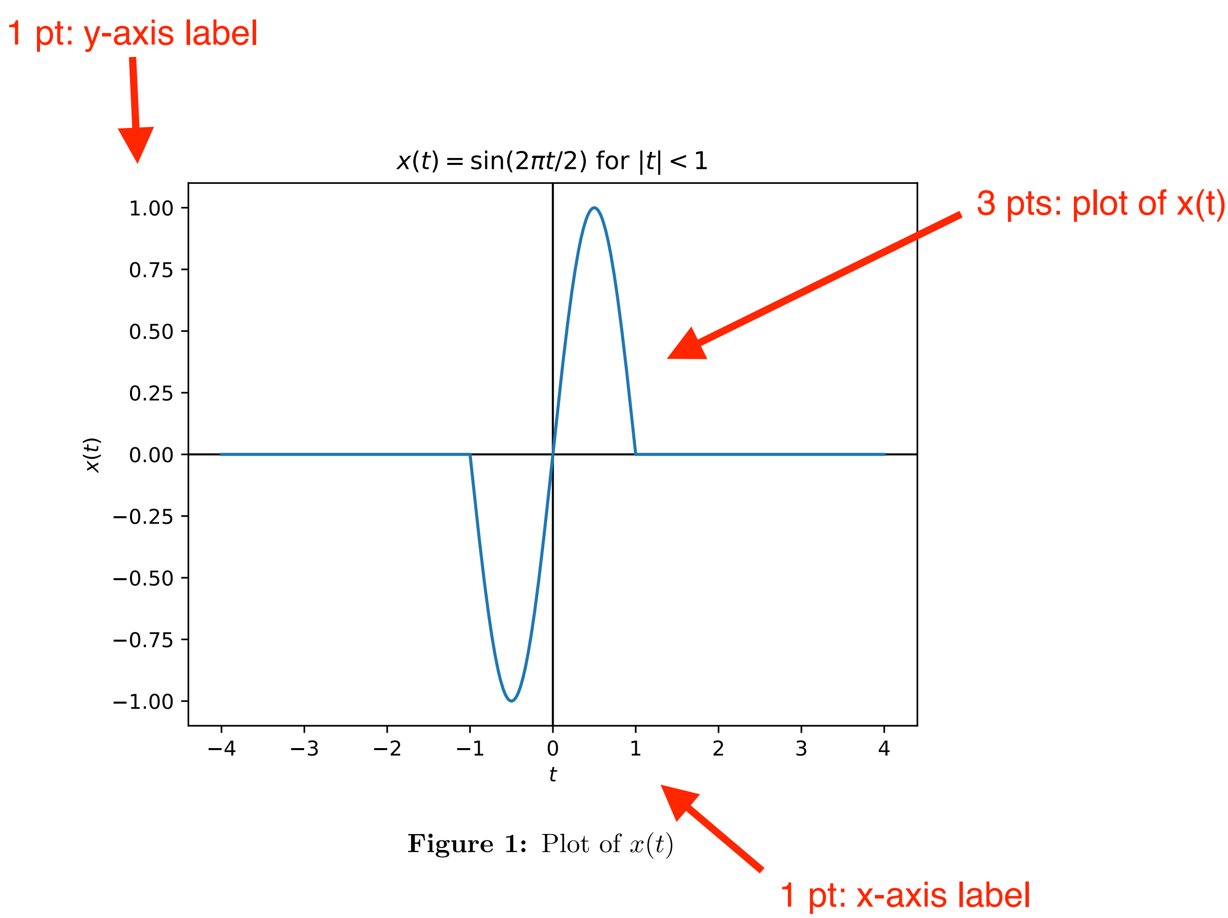


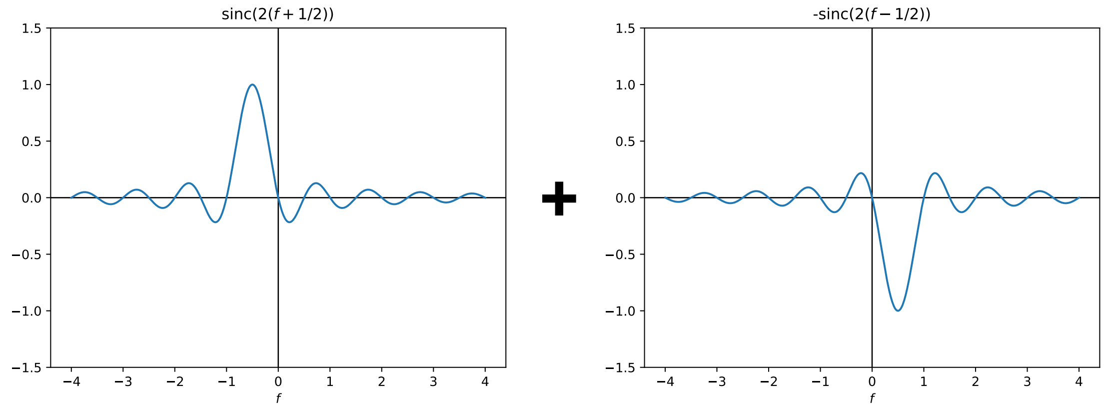
Figure 1: Plot of $x(t)$

(b)

$$
\begin{aligned}
x(t) &= \operatorname{rect}\left(\frac{t}{2}\right) \times \sin(2\pi t/2) \\
X(f) &= 2\operatorname{sinc}(2f) * \left(\frac{1}{2j}\left(\delta\left(f-\frac{1}{2}\right) - \delta\left(f+\frac{1}{2}\right)\right)\right) \\
&= -j\operatorname{sinc}(2f) * \left(\delta\left(f-\frac{1}{2}\right) - \delta\left(f+\frac{1}{2}\right)\right) \\
&= -j\left(\operatorname{sinc}\left(2\left(f-\frac{1}{2}\right)\right) - \operatorname{sinc}\left(2\left(f+\frac{1}{2}\right)\right)\right) \\
&= j\left(\operatorname{sinc}\left(2\left(f+\frac{1}{2}\right)\right) - \operatorname{sinc}\left(2\left(f-\frac{1}{2}\right)\right)\right)
\end{aligned}
$$

3 pts: a correct expression of x(t)
3 pts: CTFT of rect(t/2)
3 pts: CTFT of sin(2*pi*t/2)
3 pts: multiplication in time domain is convolution in frequency domain
3 pts: a correct final answer

(c) $X(f) = j\left(\text{sinc}\left(2\left(f + \frac{1}{2}\right)\right) - \text{sinc}\left(2\left(f - \frac{1}{2}\right)\right)\right)$



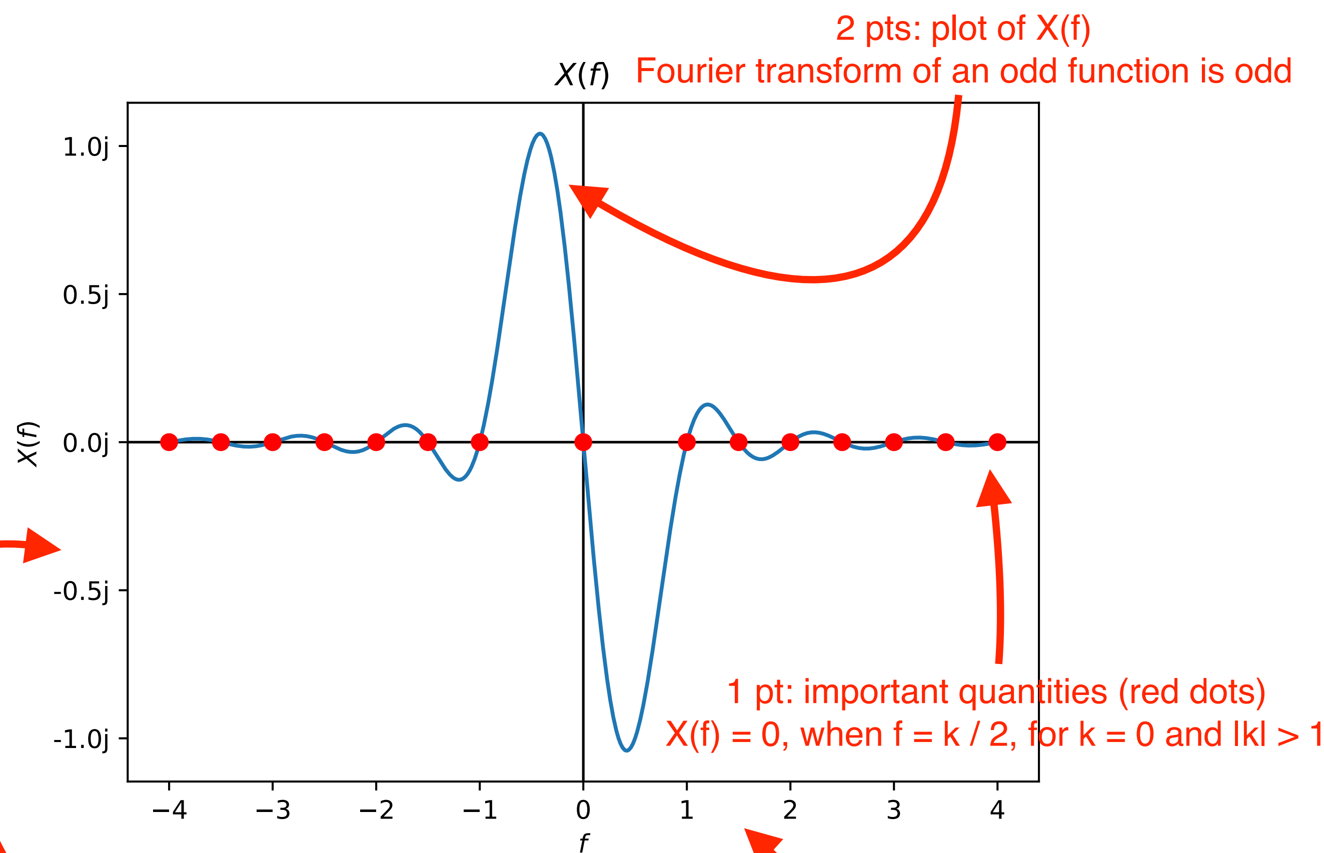


Figure 2: Plot of $X(f)$

Problem 4

Consider the real-valued continuous-time signal $x(t) = cos(2\pi(6000)t)$.
This signal is sampled at an 8 kHz rate with an ideal sampler (no prefilter; so it is not an ideal A/D) to generate the continuous-time (CT) sampled signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

where T is the sampling interval.

a. (10) Find a simple expression for the CTFT $X_s(f)$ of $x_s(t)$. Your final answer should not contain any operators, such as convolution, rep, or comb.

b. (5) Carefully sketch $X_s(f)$. Be sure to dimension all important quantities on both the horizontal and vertical axes.

Suppose we reconstruct a continuous-time signal y(t) from $x_s(t)$ by filtering it with an ideal low-pass filter with cutoff frequency $f_c = 4kHz$, and gain T in the passband.

c. (10) Find a simple expression for $y(t)$. Your final answer should not contain any operators, such as convolution, rep, or comb.

a.

$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x(nT)\sigma(t - nT) \\ &= comb_T[x(t)], \text{ where } T = \frac{1}{8kHz}, f_s = 8kHz \end{aligned}$$

Using the following CTFT transform pairs,

$$cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

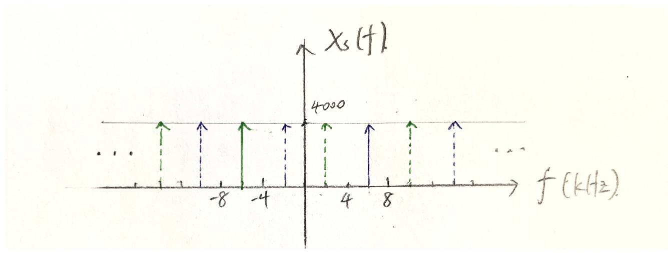
$$comb_T[x(t)] \longleftrightarrow \frac{1}{T} rep_{\frac{1}{T}}[X(f)]$$

and get

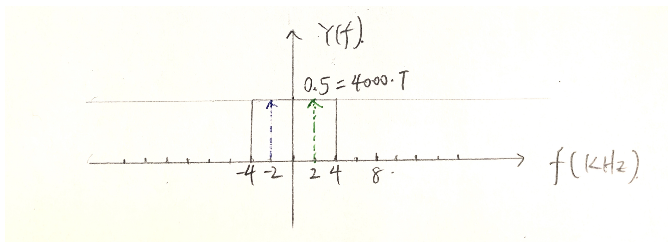
$$X(f) = \frac{1}{2}[\delta(f - 6000) + \delta(f + 6000)]$$

$$\begin{aligned} X_s(f) &= \frac{1}{T} rep_{\frac{1}{T}}\{\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]\} \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{1}{2}[\delta(f - k\frac{1}{T} - f_0) + \delta(f - k\frac{1}{T} + f_0)] \\ &= 8000 \sum_{k=-\infty}^{\infty} \frac{1}{2}[\delta(f - 8000k - 6000) + \delta(f - 8000k + 6000)] \end{aligned}$$

b.



c. After LPF, get Y(f):



$$Y(f) = T \cdot \frac{1}{T} \cdot \frac{1}{2}[\delta(f - 2000) + \delta(f + 2000)]$$
$$= \frac{1}{2}[\delta(f - 2000) + \delta(f + 2000)]$$

Do inverse CTFT to $Y(f)$, and get

$$y(t) = cos(2\pi(2000)t)$$

a.
2 pts: knowing the expression of comb operator ($x_s(t)$ is $comb_T[x(t)]$).
3 pts: knowing CTFT of $cos(2\pi f_0 t)$.(1 pt for finding correct f_0).
3 pts: knowing CTFT pair of comb and rep. (1 pt for finding correct T).
2 pt: knowing the expression of rep operator (answer not contain operators)

b.
1 pt: knowing what $\delta(f)$ looks like. (0.5 pts for indicating infinity at f=0).
2 pts: replicate in both directions (1 pt for each direction).
1 pt: x-axis label.
1 pt: y-axis label.

c.
2 pt: knowing how an ideal LPF behaves (what frequency region is kept).
2 pts: know what "gain T" means.

3 pt: expression of Y(f).
3 pt: inverse CTFT of Y(f).

any type of arithmetic error: -0.5 pts
typo, forgetting to rewrite a scalar: -0 pts.