19 April 2019 **Name:________________** **Signature: __________________**

ECE 438 **Final Exam** **Spring 2019**

- You have 120 minutes to work the following five problems, which are worth 25 points each, for a total of 125 points.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem. Please note that the problems are arranged in the order that the topics were covered during the semester, not necessarily in the order of difficulty to solve them.

1. (25 pts.) Consider a system described by the following equation

$$y[n]=\frac{1}{2}\left(x[n+1]+x[n-1]\right).$$

a. (2) Is this system causal? Why or why not?

b. (3) Is this system bounded-input-bounded-output (BIBO) stable? Why or why not?

c. (5) Find a simple expression for the frequency response $H(\omega)$ for this system.

d. (5) From your answer to part (c), determine simple expressions for the magnitude and phase of the frequency response, and sketch them as a function of ω for $-\pi \le \omega \le \pi$.

Consider the signal $x[n]=\cos\left(\frac{\pi}{4}n\right), -\infty < n < \infty$.

e. (5) Find the DTFT $X(\omega)$ of this signal.

f. (5) Use your answer to part (e) above and the frequency response $H(\omega)$ that you determined in part (c) to find a simple expression for the response $y[n]$ to the system when $x[n]$ is given as above.

2pt (a). Not causal. $y[n]$ depends on future value $x[n+1]$.

(b). Yes. 1pt For $|x[n]| \le M$, $y[n] \le \frac{1}{2}|2M| = M$. 2pt

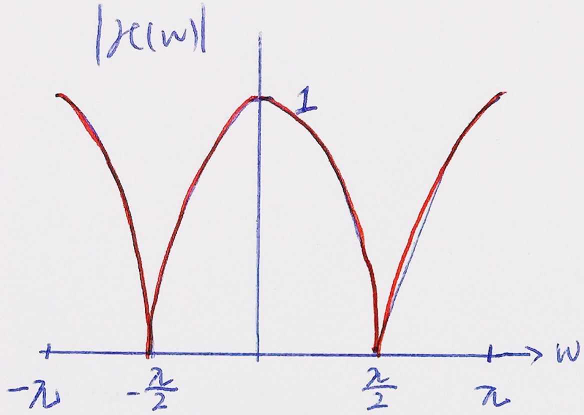
(c). $H(w) = \frac{Y(w)}{X(w)}$ 1pt $= \frac{1}{2}(e^{jw} + e^{-jw})$ 2pt $= \cos(w)$. 2pt

(d). The magnitude: $|H(w)| = |\cos(w)|$

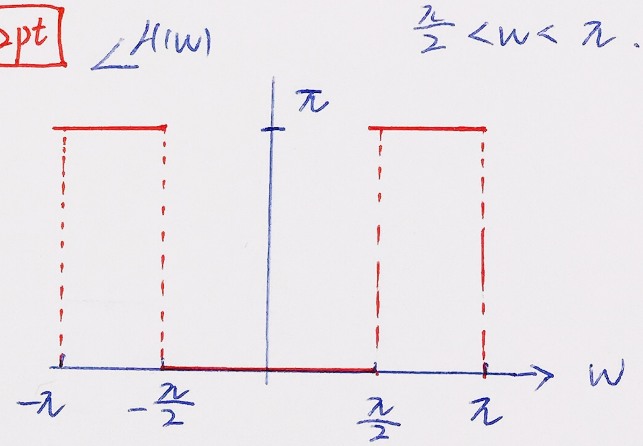
1 pt

The phase: $\angle H(w) = \angle \cos(w) = \begin{cases} 0, & -\frac{\pi}{2} < w < \frac{\pi}{2} \\ \pi, & -\pi < w < -\frac{\pi}{2} \text{ \& } \frac{\pi}{2} < w < \pi. \end{cases}$

2pt



2pt



(e). 2pt Since $\cos(w_0 n) \overset{DTFT}{\longleftrightarrow} \pi\, rep_{2\pi}[\delta(w - w_0) + \delta(w + w_0)]$

3pt $\therefore X(w) = \pi\, rep_{2\pi}[\delta(w - \frac{\pi}{4}) + \delta(w + \frac{\pi}{4})]$.

(f) 1pt $Y(w) = H(w)X(w) = \cos(w) \cdot \pi\, rep_{2\pi}[\delta(w - \frac{\pi}{4}) + \delta(w + \frac{\pi}{4})]$.

2pt $= \pi\, rep_{2\pi}[\cos\frac{\pi}{4}((\delta(w - \frac{\pi}{4}) + \delta(w + \frac{\pi}{4})]$.

$y[n] = \cos\frac{\pi}{4} \cos\frac{\pi}{4}[n]$

2pt $= \frac{\sqrt{2}}{2} \cos(\frac{n\pi}{4})$

2. (25 pts.) Consider the 21-point signal $x[n]$ defined below

$$x[n] = \begin{cases} 0, & n = 0, \\ 1, & 1 \le n \le 10, \\ -1, & 11 \le n \le 20 \end{cases}$$

And consider the 21-point DFT defined by

$$X[k] = \sum_{n=0}^{20} x[n] e^{-j2\pi kn/21}, \; k = 0,\ldots,20$$

a. (3) Compute $X[0]$, and comment on the relationship between its value and the structure of $x[n]$.

b. (16) Find a simple analytic expression for $X[k]$ for all k. Be sure to evaluate all summations, and simplify your answer as much as possible.

c. (6) Approximately sketch $|X[k]|$, $k = 0,\ldots,20$.

Problem 2: 21-point DFT signal $x[n]$:

$$x[n] = \begin{cases} 0, & n=0 \\ 1, & 1 \le n \le 10 \\ -1, & 11 \le n \le 20 \end{cases}$$

(a) Compute $X[0]$. Comment Relationship between $X[0]$ and $x[n]$

Soln: Using the definition of DFT:

$$X[k] = \sum_{n=0}^{20} x[n] e^{-j2\pi kn/21}, \quad k=0, \cdots 20$$

when $k=0$

$$X[k] = \sum_{n=0}^{20} x[n] e^{-j2\pi \cdot 0 \cdot n/21} = \sum_{n=0}^{20} x[n] = 0$$ −2pt

In frequency domain, when $k=0$. $X[0]=0 \Rightarrow$ DC component of $x[n]$ is 0

−1pt

(2) Find DFT expression $X[k]$ for all k

Soln: We know that using the DFT pairs; calculating N-pt DFT

when $x[n] = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{else} \end{cases}, \quad 0 \le n \le N-1$

$$X(k) = e^{-j\frac{2\pi k}{N}(M-1)/2} \frac{\sin[2\pi kM/(2N)]}{\sin[2\pi k/(2N)]}$$ [Using relation DTFT] ① −3pt

Also $x[n-n_0] \xrightarrow{DFT} X[k] e^{-j2\pi k n_0/N}$ ② −3pt

In our case $x[n] = (u[n-1] - u[n-11]) - (u[n-11] - u[n-21])$

Let $x_1[n] = u[n] - u[n-10]$

using ① DFT of $x_1[n]$ $X_1[k] = e^{-j\frac{2\pi k}{21} \cdot \frac{9}{2}} \frac{\sin[2\pi k \cdot 10/(2 \cdot 21)]}{\sin[2\pi k/(2 \cdot 21)]}$

$$= e^{-j\frac{9}{21}\pi k} \frac{\sin(\frac{10}{21}\pi k)}{\sin(\frac{1}{21}\pi k)}$$ −3pt

Because $x_1[n-1] = u[n-1] - u[n-11]$

$\therefore$ DFT of $u[n-1] - u[n-11] \xrightarrow{DFT} X_1[k] e^{-j\frac{2\pi k \cdot 1}{21}}$

$$= e^{-j\frac{11}{21}k} \left(\frac{\sin\left(\frac{10}{21}\pi k\right)}{\sin\left(\frac{1}{21}\pi k\right)} \right) \quad ③ \quad -3pt$$

We could do the same thing for $u[n-11] - u[n-21]$

Let $x_2[n] = u[n] - u[n-10]$

$x_2[n-11] = u[n-11] - u[n-21]$

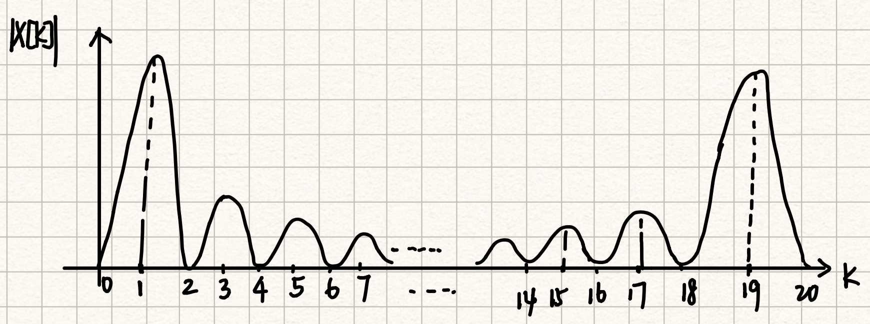
$$x_2[n-11] \xrightarrow{DFT} e^{-j\frac{9}{21}\pi k} \left(\frac{\sin\left(\frac{10}{21}\pi k\right)}{\sin\left(\frac{1}{21}\pi k\right)} \right) e^{-j\frac{2\pi}{21}\cdot 11k}$$

$$= e^{-j\frac{31}{21}\pi k} \left(\frac{\sin\left(\frac{10}{21}\pi k\right)}{\sin\left(\frac{1}{21}\pi k\right)} \right) \quad ④ \quad -1pt$$

Combine ③.④ $X[k] = \left(\frac{\sin\left(\frac{10}{21}\pi k\right)}{\sin\left(\frac{1}{21}\pi k\right)} \right) \left(e^{-j\frac{11}{21}\pi k} - e^{-j\frac{31}{21}\pi k} \right)$ $-3pt$

(If students write out using the Linearity property of DFT. 2pt will be added)

(3) Approximately sketch $|X[k]|$

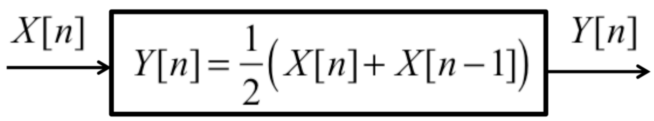


$-6pt$

3. (25 pts.) Let $X[n]$ be a wide-sense stationary sequence of random variables with zero mean and autocorrelation function

$$r_{XX}[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & |n| = 1 \\ 0, & \text{else} \end{cases}$$

Suppose that this sequence is processed with the following filter to generate the output sequence $Y[n]$:



a. (5) Find the mean of the sequence $Y[n]$.

b. (10) Find the cross-correlation $r_{XY}[n]$ between $X[n]$ and $Y[n]$.

c. (10) Find the auto-correlation $r_{YY}[n]$ of the output $Y[n]$.

Question 3

$X[n]$ is a wide sense stationary sequence of random variables: (1) $\mathbb{E}\big[X[n]\big]$ is a constant. (2) auto-correlation function is a function of the time difference

Rewrite $r_{XX}[n]$ in another form

$$r_{XX}[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]$$

Since $Y[n] = \frac{1}{2}(X[n] + X[n-1])$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

(a). **Find the mean of the sequence $Y[n]$**

$$\mathbb{E}\big[Y[n]\big] = \mathbb{E}\big[\frac{1}{2}\big(X[n] + X[n-1]\big)\big]$$

2 pts $= \frac{1}{2}\mathbb{E}\big[X[n]\big] + \frac{1}{2}\mathbb{E}\big[X[n-1]\big]$

1 pts since $X[n]$ is wide sense stationary, $\mathbb{E}\big[X[n]\big]$ is independent of n
also, $X[n]$ has zero mean

2 pts $= 0$

(b). **Find the cross-correlation $r_{XY}[n]$ between $X[n]$ and $Y[n]$**

10 pts Method 1

$$
\begin{aligned}
r_{XY}[n] &= \mathbb{E}\big[X[m]Y[m+n]\big] \quad \text{2 pts} \\
&= \mathbb{E}\big[X[m]\frac{1}{2}\big(X[m+n] + X[m+n-1]\big)\big] \\
&= \frac{1}{2}\mathbb{E}\big[X[m]X[m+n]\big] + \frac{1}{2}\mathbb{E}\big[X[m]X[m+n-1]\big] \\
&= \frac{1}{2}r_{XX}[n] + \frac{1}{2}r_{XX}[n-1] \quad \text{2 pts} \\
&= \frac{1}{2}\left(\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]\right) + \frac{1}{2}\left(\delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n]\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\delta[n+1] + \frac{3}{2}\delta[n] + \frac{3}{2}\delta[n-1] + \frac{1}{2}\delta[n-2]\right) \\
&= \frac{1}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] \quad \text{2 pts}
\end{aligned}
$$

Remaining steps: 4 pts

10 pts Method 2

2 pts $r_{XY}[n] = r_{XX}[n] * h[n]$ 2 pts 2 pts

$$= \left(\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]\right) * \left(\frac{1}{2}(\delta[n] + \delta[n-1])\right)$$

Remaining steps: 2 pts

$$= \frac{1}{2}\left(\delta[n] + \delta[n-1] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n]\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\delta[n+1] + \frac{3}{2}\delta[n] + \frac{3}{2}\delta[n-1] + \frac{1}{2}\delta[n-2]\right)$$

$$= \frac{1}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2]$$ 2 pts

(c). **Find the auto-correlation $r_{YY}[n]$ of the output $Y[n]$**

10 pts Method 1

2 pts $r_{YY}[n] = \mathbb{E}\big[Y[m]Y[m+n]\big]$

$$= \mathbb{E}\big[\frac{1}{2}(X[m] + X[m-1])\frac{1}{2}(X[m+n] + X[m+n-1])\big]$$

$$= \frac{1}{4}\mathbb{E}\big[X[m]X[m+n] + X[m]X[m+n-1] + X[m-1]X[m+n] + X[m-1]X[m+n-1]\big]$$

$$= \frac{1}{4}\left(r_{XX}[n] + r_{XX}[n-1] + r_{XX}[n+1] + r_{XX}[n]\right)$$

$$= \frac{1}{8}\delta[n+2] + \frac{1}{2}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{8}\delta[n-2]$$ 2 pts

Remaining steps: 6 pts

10 pts Method 2

2 pts $r_{YY}[n] = h[n] * r_{XY}[-n]$

$$= \left(\frac{1}{2}(\delta[n] + \delta[n-1])\right) * \left(\frac{1}{4}\delta[-n+1] + \frac{3}{4}\delta[-n] + \frac{3}{4}\delta[-n-1] + \frac{1}{4}\delta[-n-2]\right)$$

$$= \left(\frac{1}{2}(\delta[n] + \delta[n-1])\right) * \left(\frac{1}{4}\delta[n-1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n+1] + \frac{1}{4}\delta[n+2]\right)$$

$$= \frac{1}{8}\delta[n-1] + \frac{3}{8}\delta[n] + \frac{3}{8}\delta[n+1] + \frac{1}{8}\delta[n+2] + \frac{1}{8}\delta[n-2] + \frac{3}{8}\delta[n-1] + \frac{3}{8}\delta[n] + \frac{1}{8}\delta[n+1]$$

$$= \frac{1}{8}\delta[n+2] + \frac{1}{2}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{8}\delta[n-2]$$ 2 pts

Remaining steps: 6 pts

4. (25 pts) The 2-D signal $f(x,y) = 1 + \cos\left(2\pi(90x + 10y)\right)$ is sampled with an ideal sampler at 100 dpi in both x and the y directions to generate the signal $f_s(x,y)$. The units of x and y are inches.

   a. (3) Sketch $f(x,y)$ with sufficient detail and accuracy to indicate that you know what it looks like.

   b. (3) Find a simple expression for the CSFT $F(u,v)$ of $f(x,y)$.

   c. (3) Find a simple expression for $f_s(x,y)$ using standard functions and operators that were defined in class.

   d. (3) Find a simple expression for the CSFT $F_s(u,v)$ of $f_s(x,y)$. Your answer should not contain any operators.

   e. (3) Sketch what the spectrum $F_s(u,v)$ would look like. Be sure to dimension all important quantities, including units for the dimensions.

   You reconstruct the continuous-parameter signal $f_r(x,y)$ by filtering $f_s(x,y)$ with a linear, shift-invariant filter having point spread function $\text{sinc}(100x, 100y)$.

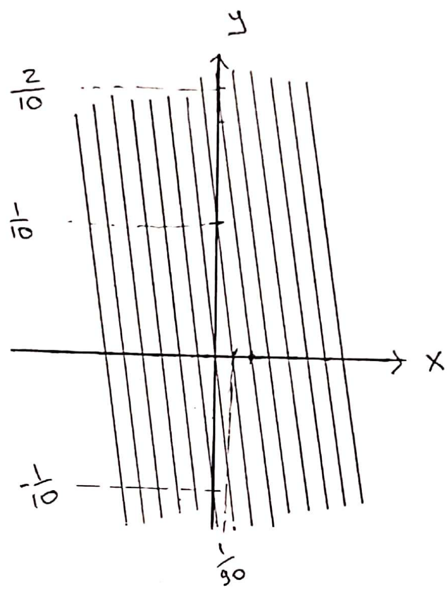
   f. (5) Find a simple expression for the reconstructed signal $f_r(x,y)$.

   g. (5) Sketch $f_r(x,y)$ with sufficient detail and accuracy to indicate that you know what it looks like.

Problem 4

$$f(x, y) = 1 + \cos(2\pi(90x + 10y))$$

a. Sketch $f(x, y)$ Contours show where the value of function $f(x, y) = 2$. This is found by the following equation: $2\pi(90x + 10y) = 2\pi k$



3 pts

b. Find a simple expression for the CSFT $F(u, v)$
Using CSFT pairs:

$$F(u, v) = \delta(u, v) + \frac{1}{2}[\delta(u - 90, v - 10) + \delta(u + 90, v + 10)]$$

3 pts

c. Find a simple expression for $f_s(x, y)$ using standard functions and operators that were defined in class.

$$\begin{aligned} f_s(x, y) &= \text{comb}_{1/100,1/100} f(x, y) \\ &= \text{comb}_{1/100,1/100}[1 + \cos(2\pi(90x + 10y))] \end{aligned}$$

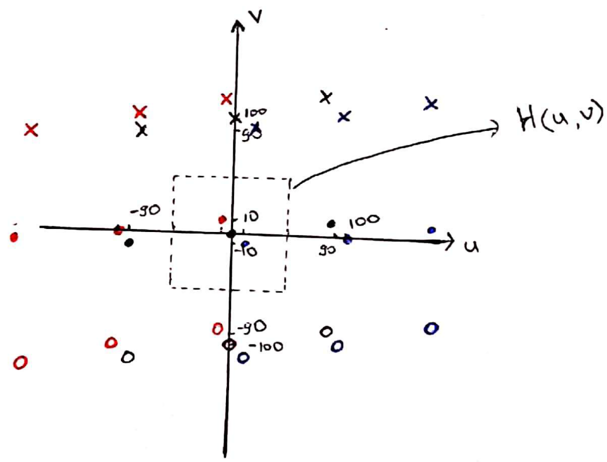
3 pts

d. Find a simple expression for the CSFT $F_s(u, v)$ of $f_s(x, y)$. Your answer should not contain any operators.

$$\begin{aligned} F_s(u, v) &= 100 \times 100 \times \text{rep}_{100,100}[F(u, v)] \\ &= 10000\text{rep}_{100,100}[\delta(u, v) + \frac{1}{2}[\delta(u - 90, v - 10) + \delta(u + 90, v + 10)]] \\ &= 10000 \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Big[\delta(u - 100k, v - 100m) \\ &+ \frac{1}{2}[\delta(u - 90 - 100k, v - 10 - 100m) + \delta(u + 90 - 100k, v + 10 - 100m)]\Big] \end{aligned}$$

3 pts

e. Sketch what the spectrum $F_s(u,v)$ would look like. Be sure to dimension all important quantities, including units for the dimensions.
The different colors and different markers show different repetitions of the original spectrum $F(u,v)$.



3 pts

f. Find a simple expression for the reconstructed signal $f_r(x,y)$

$$f_r(x,y) = f_s(x,y) ** sinc(100x, 100y)$$

$$F_r(u,v) = F_s(u,v) \times \frac{1}{10000} rect(\frac{u}{100}, \frac{v}{100})$$

$$= \delta(u,v) + \frac{1}{2}\left[\delta(u+10, v-10) + \delta(u-10, v+10)\right] \text{ (\textbf{found graphically})}$$

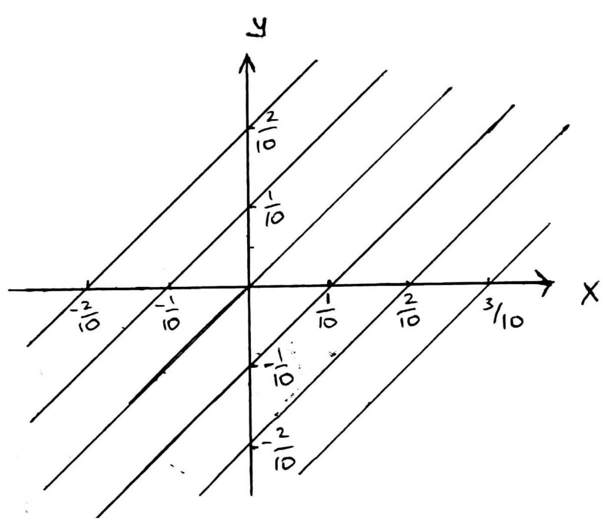
3 pts

Then, using inverse CSFT pairs:

$$f_r(x,y) = 1 + \cos(2\pi(10x - 10y))$$

2 pts

g. Sketch $f_r(x,y)$ with sufficient detail and accuracy to indicate that you know what it looks like.



5 pts

5. (25 pts) Consider a spatial filter with *nonseparable* point spread function $h[m,n]$ given below

	$h[m,n]$	$n = -1$	0	1
	-1	$\frac{1}{3}$	0	$-\frac{1}{3}$
m	0	0	1	0
	1	$-\frac{1}{3}$	0	$\frac{1}{3}$

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 7×7 set of pixels in the input image.

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	1	1	0	0
0	1	1	1	1	1	0
0	1	1	1	1	1	1
0	1	1	1	1	1	1

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, the $\mu = \nu$, and the $\mu = -\nu$ axis.

c. (3) Using your results from parts (a) and (b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

(a)

Extended input:

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	1	1	1	0	0	0
0	0	1	1	1	1	1	0	0
0	0	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1

Output: (10pt)

0	0	0	0	0	0	0
0	0	1/3	0	-1/3	0	0
0	1/3	1/3	1	-1/3	-1/3	0
1/3	1/3	2/3	1	4/3	-1/3	-1/3
1/3	1	2/3	1	4/3	4/3	0
0	1	1	1	1	4/3	4/3
0	1	1	1	1	1	1

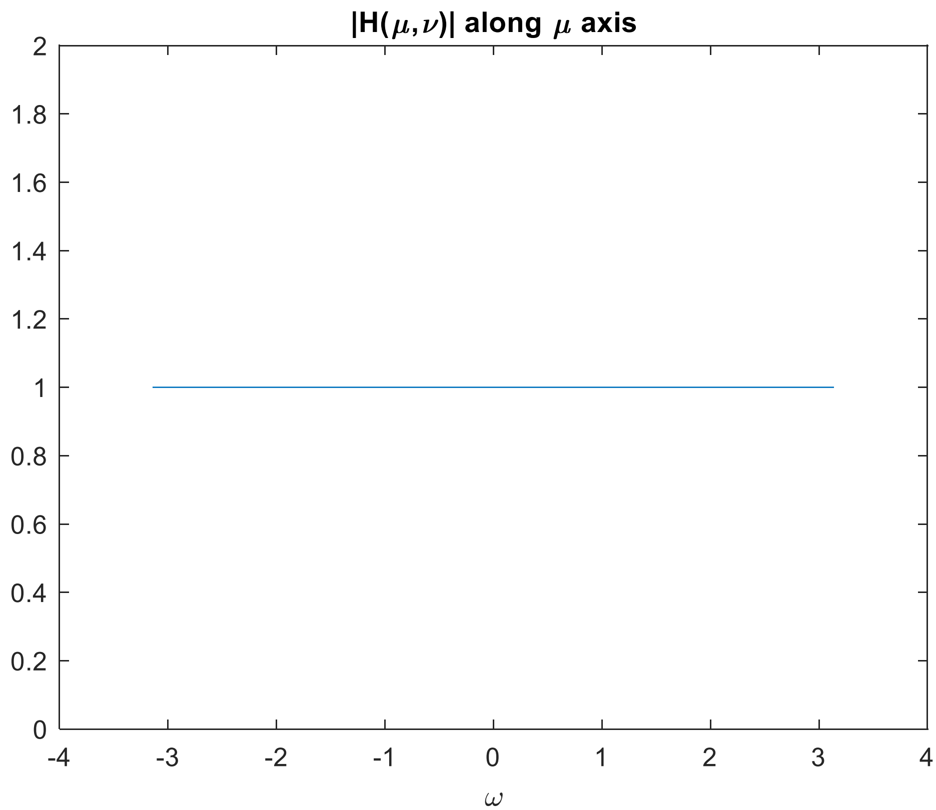
(b)

$$h[m,n] = \frac{1}{3}\delta[m+1,n+1] + \frac{1}{3}\delta[m-1,n-1] - \frac{1}{3}\delta[m+1,n-1] - \frac{1}{3}\delta[m-1,n-1] + \delta[m,n]$$

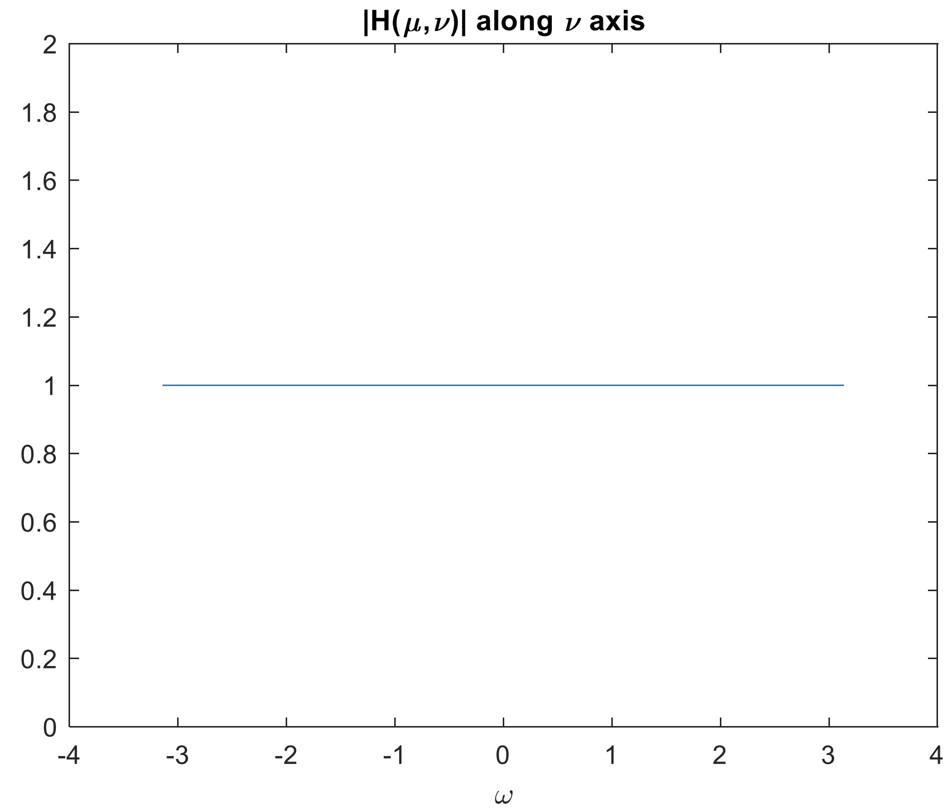
$$\Rightarrow H(\mu,\nu) = \frac{1}{3}e^{j(\mu+\nu)} + \frac{1}{3}e^{-j(\mu+\nu)} - \frac{1}{3}e^{j(\mu-\nu)} - \frac{1}{3}e^{-j(\mu-\nu)} + 1 = \frac{2}{3}(\cos(\mu+\nu) - \cos(\mu-\nu)) + 1$$

because $\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2}\right)\sin(\frac{y-x}{2})$, $H(\mu,\nu)$ can also be written as

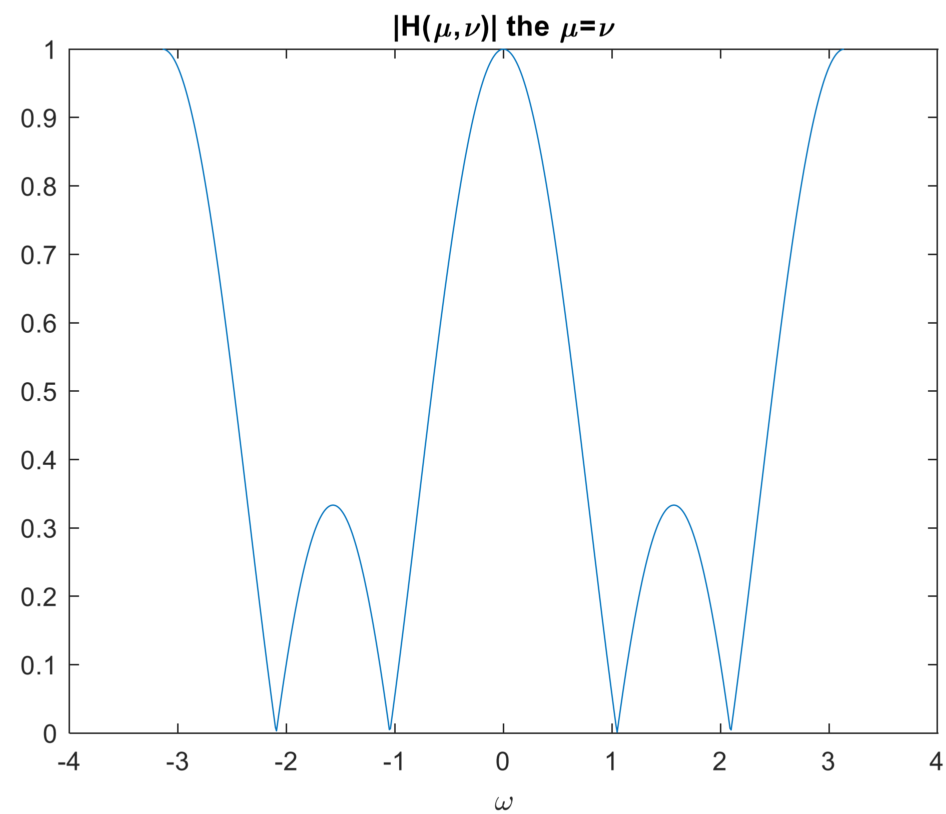
$H(\mu,\nu) = \frac{-4}{3}\sin(\mu)\sin(\nu) + 1$ **(2pt)**



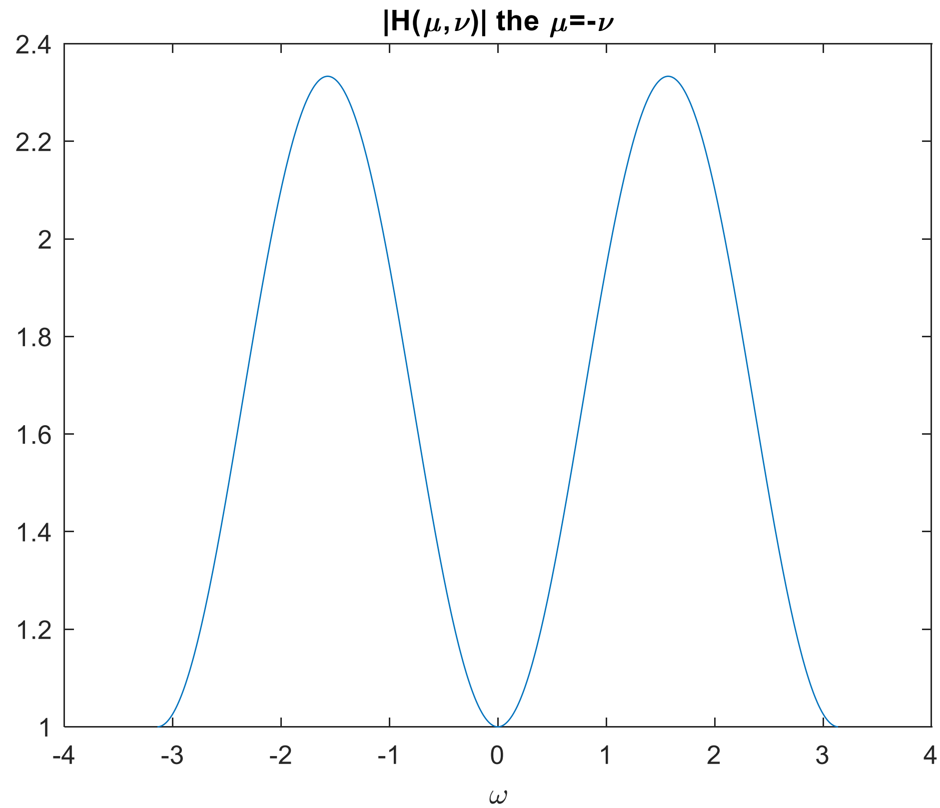
2pt



2pt



3pt



3pt

(c) **3pt**

The filter doesn't affect the frequency along the μ axis and ν axis.

When $\mu = \nu$, the mid-frequency around $\frac{\pi}{2}$ is suppressed and when $\mu = -\nu$, the mid-frequency is enhanced. The edge is emphasized.

Since the DC response is 1, the center region of 1 is maintained.