17 April 2019 **Name:____________________**

ECE 438 **Exam No. 3** **Spring 2019**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider two random variables X and Y which are jointly distributed according to the following bivariate density function

$$f_{XY}(x,y)=\begin{cases} 2, & 0\le y\le x, 0\le x\le 1, \\ 0, & \text{else} \end{cases}$$

a. (5) Sketch $f_{XY}(x,y)$.

b. (6) Find and sketch the marginal densities $f_X(x)$ and $f_Y(y)$.

c. (2) Are X and Y independent?

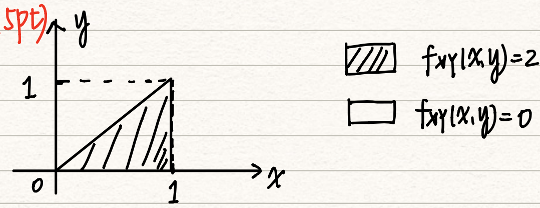
d. (12) Find the mean and variance of X and Y and the correlation coefficient ρ_{XY} between them.

Problem 1:

$$f_{XY}(x,y) = \begin{cases} 2, & 0 \le y \le x,\ 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

① Sketch $f_{XY}(x,y)$: (5pt)

- 5pt

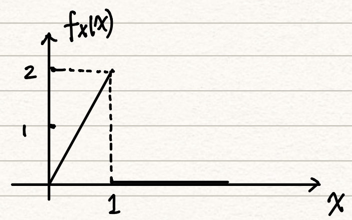


② Find and sketch $f_X(x)$ $f_Y(y)$ (6pt)

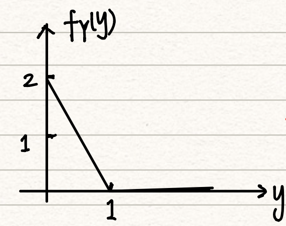
$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_0^x 2\,dy = 2x \quad 0 \le x \le 1$ - 2pt

$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_y^1 2\,dx = 2(1-y) \quad 0 \le y \le 1$ - 2pt

$\therefore\ f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2(1-y), & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$



- 1pt



- 1pt

③ X.Y independent? (2pt)

No! X.Y are not independent
Because $f_X(x) f_Y(y) \neq f_{XY}(x,y)$ - 2pt

④ Mean & Variance of X.Y. Also ρ_{XY}? (12pt)

$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$ - 2pt

$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{2}$ - 1pt

$Var(X) = \sigma_X^2 = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$ - 2pt

$E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_0^1 2y(1-y)\,dy = y^2 - \frac{2}{3}y^3 \Big|_0^1 = \frac{1}{3}$ - 2pt

$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \int_0^1 2y^2(1-y)\,dy = \frac{2}{3}y^3 - \frac{1}{2}y^4 \Big|_0^1 = \frac{1}{6}$ - 1pt

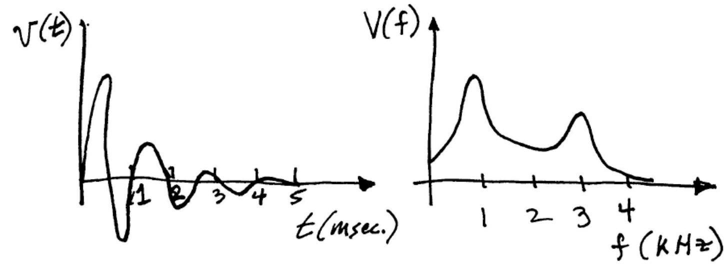
$Var(Y) = \sigma_Y^2 = E[Y^2] - E^2[Y] = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$ - 2pt

$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{XY}(x,y)\,dx\,dy = \int_0^1 x\,dx \int_0^x 2y\,dy = \int_0^1 x^3\,dx = \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{4}$ - 1pt

$$\therefore \rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{\frac{1}{4} - \frac{2}{3} \times \frac{1}{3}}{\sqrt{\frac{1}{18}} \cdot \sqrt{\frac{1}{18}}} = \frac{\frac{1}{36}}{\frac{1}{18}} = \frac{1}{2}$$

- 1pt

2. (25) Consider a voiced phoneme for which the time-domain, continuous-time vocal tract response $v(t)$ and corresponding vocal tract frequency response (CTFT) $V(f)$ are given below.



a. (8) Assume that the pitch frequency for the speaker is 100 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

b. (8) For your speech waveform $s(t)$ in part (a), sketch what the CTFT $S(f)$ would look like. Be sure to dimension all important quantities in $S(f)$.

c. (9) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length of 50 samples. Carefully sketch the resulting spectrogram as a function of the discrete-time index n and digital frequency ω (radians/sample). Be sure to dimension all important quantities in your plot. Is this a wide-band or a narrow-band spectrogram?

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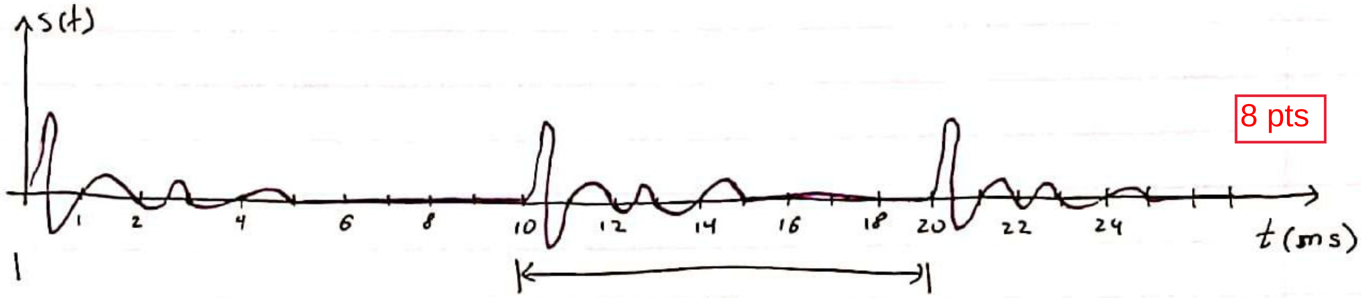
Q. 2

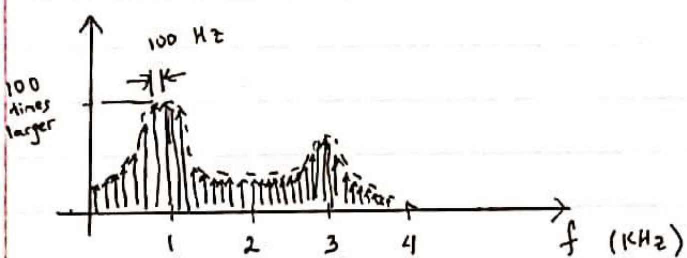
a) Pitch Period $= \frac{1}{\text{Pitch Frequency}} = \frac{1}{100} = 10\,ms$

$s(t) = v(t) * e(t)$ $e(t) = rep_{10ms}[\delta(t)] = rep_{0.01}[\delta(t)]$



8 pts

b) $S(f) = V(f) \cdot E(f)$ $E(f) = \frac{1}{0.01} comb_{\frac{1}{0.01}}[1] = 100\ comb_{100}[1]$



8 pts

c) From $V(f)$ two formant frequencies are: $F_1 = 1\,KHz$ $F_2 = 3\,KHz$

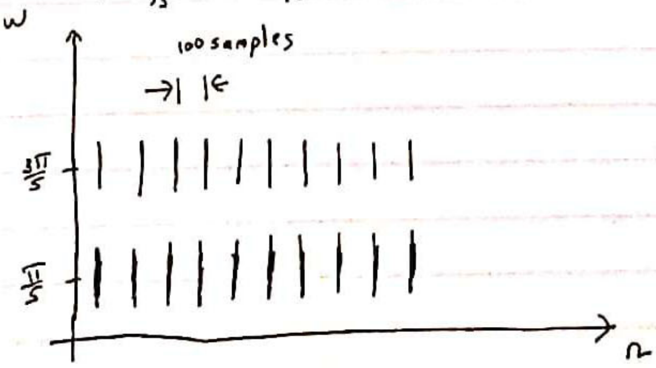
Pitch period in samples is $10ms \cdot 10\,KHz = 100$ samples

StDTFT window size is 50 samples < pitch period

↳ wide band spectrogram

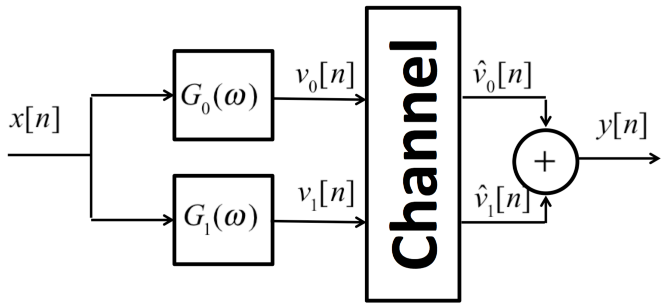
$w_1 = F_1 \cdot \frac{2\pi}{f_s} = 1 \cdot \frac{2\pi}{10} = \frac{\pi}{5}$ $w_2 = \frac{3\pi}{5}$

4 pts



5 pts

3. (25) Consider the two-channel filter bank shown below:



We will consider the channel to be ideal; so $\hat{v}_0[n] \equiv v_0[n]$ and $\hat{v}_1[n] \equiv v_1[n]$.

Suppose that the two analysis filters $G_0(\omega)$ and $G_1(\omega)$ are defined by the following two time-domain equations, respectively:

$$v_0[n] = x[n] + x[n-1] \quad v_1[n] = x[n] - x[n-1].$$

a. (10) Find simple expressions for the magnitude of the frequency responses $|G_0(\omega)|$ and $|G_1(\omega)|$, and sketch them for $-\pi \le \omega \le \pi$.

Suppose that the input to the system is given by

$$x[n] = \begin{cases} 5-n, & 0 \le n \le 5 \\ 0, & \text{else} \end{cases}$$

b. (1) Sketch or tabulate $x[n]$ for $0 \le n \le 6$.

c. (6) Sketch or tabulate the signals $v_0[n]$ and $v_1[n]$ for $0 \le n \le 6$

d. (4) Sketch or tabulate the signal $y[n]$ for $0 \le n \le 6$.

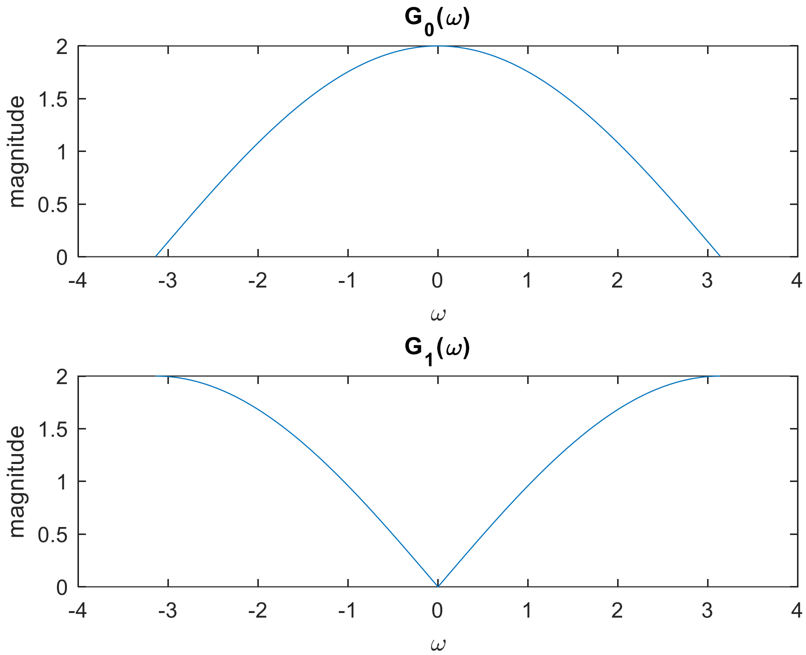
e. (1) Discuss the relationship between $x[n]$ and $y[n]$.

f. (3) Discuss the difference between the upper and lower branches of this filter bank in terms of frequency selectivity.

(a)

$$G_0(\omega) = 1 + e^{-j\omega} \Longrightarrow |G_0(\omega)| = |1 + \cos\omega - j\sin\omega| = \sqrt{(1+\cos\omega)^2 + (-\sin\omega)^2} = \sqrt{2 + 2\cos\omega}$$

$$G_1(\omega) = 1 - e^{-j\omega} \Longrightarrow |G_1(\omega)| = |1 - \cos\omega + j\sin\omega| = \sqrt{(1-\cos\omega)^2 + (\sin\omega)^2} = \sqrt{2 - 2\cos\omega}$$

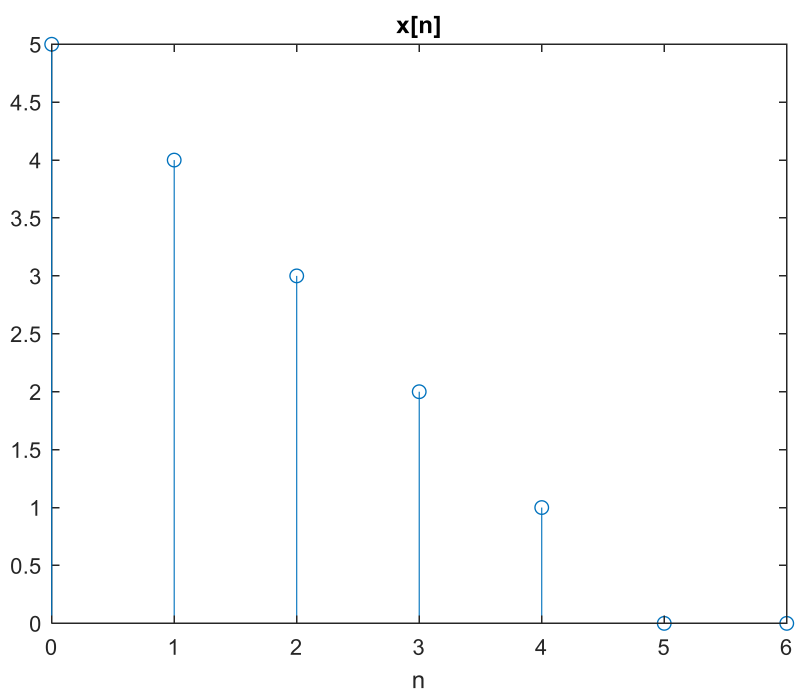


The preferred way to solve this problem is to factor out the half angle, as shown below.

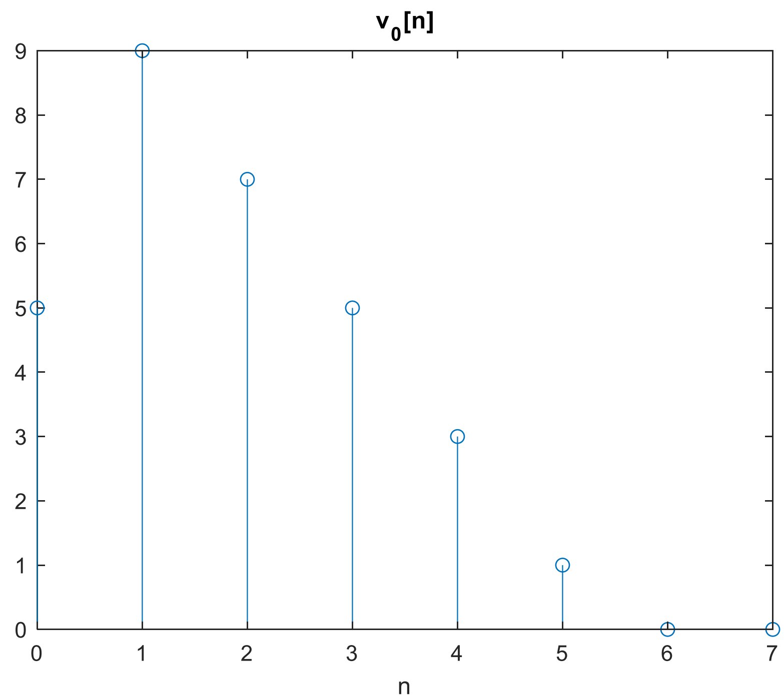
$$\begin{aligned} G_0(\omega) &= 1 + e^{-j\omega} \\ &= \left(e^{+j\omega/2} + e^{-j\omega/2}\right)e^{-j\omega/2} \\ &= 2\cos(\omega/2)e^{-j\omega/2} \end{aligned}$$

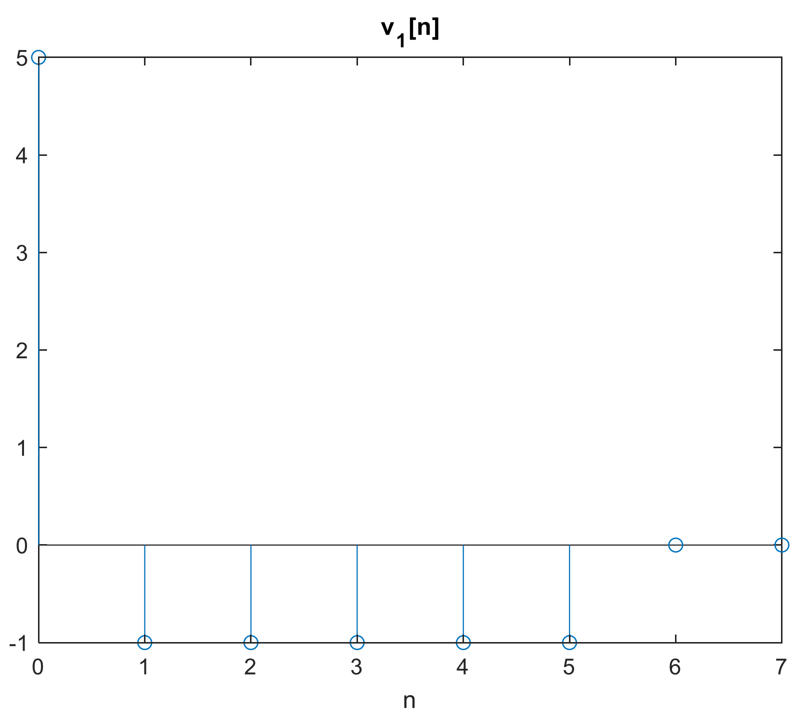
$$\left|G_0(\omega)\right| = 2\left|\cos(\omega/2)\right|$$

(b)

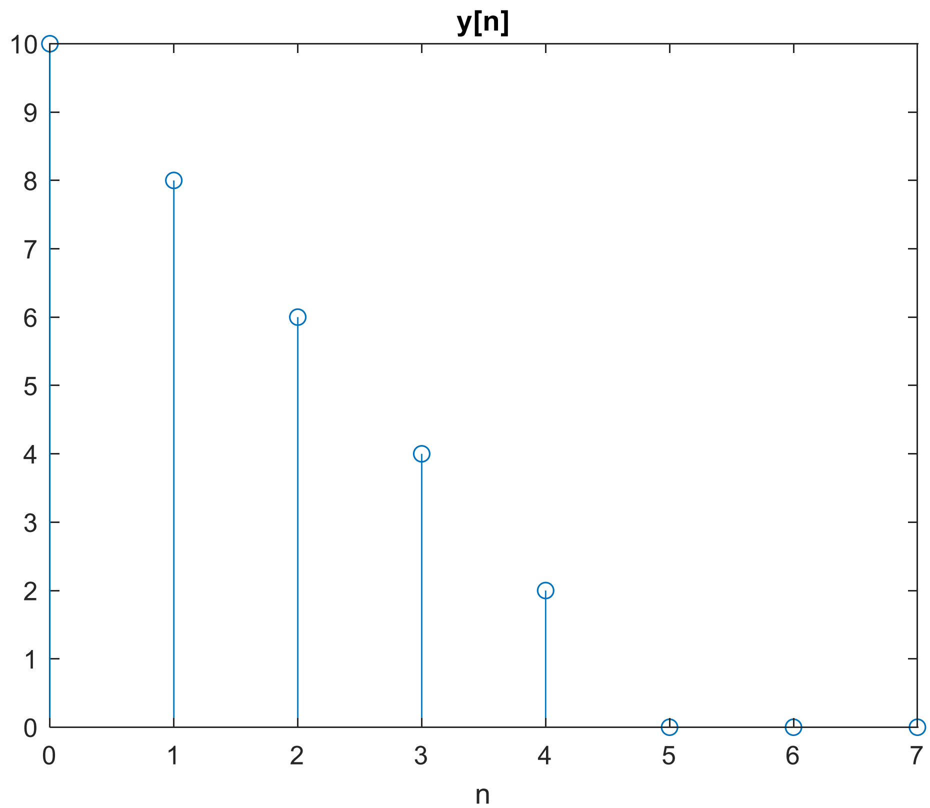


(c)





(d)



(e)

$$y[n] = 2x[n]$$

(f)

Based on the following magnitude of frequency response of these filters, we can see that the upper branch are low pass filter and the lower branch are high pass filter.

4. (25)

Consider the signal $x(t)$ s defined as follows

$$x(t) = t^{1/2}, 0 \le t \le 1$$

We wish to approximate $x(t)$ over the interval $0 \le t \le 1$ by the signal $\hat{x}(t)$ given by

$$\hat{x}(t) = a + bt \, ,$$

where a and b are constants chosen to minimize

$$\varepsilon = \int_0^1 \left[\hat{x}(t) - x(t) \right]^2 dt \, .$$

a. (19) Determine the values for a and b that will minimize ε.

b. (6) Carefully sketch $x(t)$ and $\hat{x}(t)$ on the same axes for the optimal values for a and b that you determined in part (a) above.

Exam-3 Solution

Problem 4

It would be preferable to differentiate (and use the chain rule) before multiplying out the squared term.

(a). $\varepsilon = \int_0^1 (a + bt - t^{\frac{1}{2}})^2 \, dt$ 1 pt

$= \int_0^1 (a^2 + b^2 t^2 + t + 2abt - 2at^{\frac{1}{2}} - 2bt^{\frac{3}{2}}) \, dt.$ 2 pt

$\frac{\partial \varepsilon}{\partial a} = \int_0^1 (2a + 2bt - 2t^{\frac{1}{2}}) \, dt$ 2 pt

$= 2a + b - \frac{4}{3} = 0$ 2 pt

$\frac{\partial \varepsilon}{\partial b} = \int_0^1 (2bt^2 + 2at - 2t^{\frac{3}{2}}) \, dt$ 2 pt

$= \frac{2}{3} b + a - \frac{4}{5} = 0$ 2 pt

$\Rightarrow \begin{cases} a = \frac{4}{15} & \text{4 pt} \\ b = \frac{4}{5} & \text{4 pt} \end{cases}$

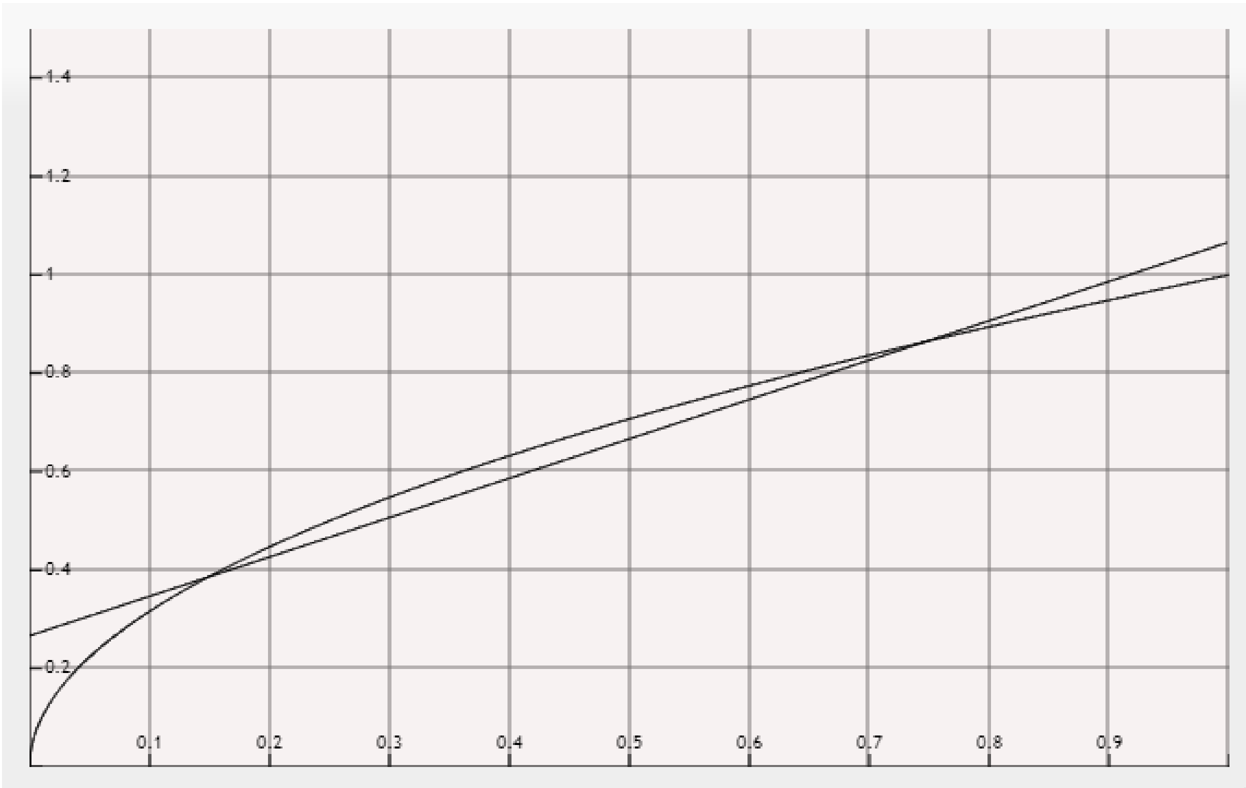
(b). $x(t) = t^{\frac{1}{2}}, \quad 0 \le t \le 1$

$\hat{x}(t) = \frac{4}{15} + \frac{4}{5} t.$

Pick value when $t = 0, \frac{1}{4}, 1.$

t	0	$\frac{1}{4}$	1
$x(t)$	0	$\frac{1}{2}$	1
$\hat{x}(t)$	$\frac{4}{15}$	$\frac{7}{15}$	$\frac{16}{15}$

6 pt



All correct / small typos. -- All points.
Minor mistakes. -- > -1 point
Some mistakes. --> -2 points
Major mistakes. --> -3 points
Do something but none of them is correct. --> will get 1- 2 point
Do not do anything. ---> 0 point