20 March 2019 **Name:____________________**

ECE 438 **Exam No. 2** Solution **Spring 2019**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts) Consider the causal DT system described by the difference equation

$$y[n] = x[n] - \tfrac{1}{2} y[n-1]$$

Use Z-transform techniques, including a partial fraction expansion, to find the response of this system to the input $x[n] = u[n]$ the unit step function.

Q1:

$$Y(z) = X(z) - \frac{1}{2} Y(z) z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + \frac{1}{2} z^{-1}} \quad (\text{Roc: } |z| > \tfrac{1}{2})$$

$$X(z) = \frac{1}{1 - z^{-1}} \quad (\text{Roc: } |z| > 1)$$

$$Y(z) = X(z) H(z) = \frac{1}{(1 - z^{-1})} \frac{1}{(1 - (-\frac{1}{2}) z^{-1})}$$

$$= \frac{A}{1 - z^{-1}} + \frac{B}{1 - (-\frac{1}{2}) z^{-1}}$$

Then: $A + \frac{A}{2} z^{-1} + B - B z^{-1} = 1$

$$\begin{cases} A + B = 1 \\ \frac{A}{2} - B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = \frac{1}{3} \end{cases}$$

$$Y(z) = \frac{\frac{2}{3}}{1 - z^{-1}} + \frac{\frac{1}{3}}{1 - (-\frac{1}{2}) z^{-1}} \quad (\text{Roc: } |z| > 1)$$

$$y[n] = \frac{2}{3} u[n] + \frac{1}{3} (-\tfrac{1}{2})^n u[n].$$

2. (25 pts)

a) (5) Let $x[n] = e^{j2\pi(2)n/8}$, $n = 0,1,2,3,4,5,6,7$. Find the 8-point DFT $X[k], k = 0,1,2,3,4,5,6,7$ of $x[n]$.

b) (2) Carefully, **but approximately,** sketch $X[k], k = 0,1,2,3,4,5,6,7$ from your answer to part (a) above. In your sketch, you may ignore any linear phase factors.

c) (12) Let $x[n] = e^{j2\pi(5)n/16}$, $n = 0,1,2,3,4,5,6,7$. Find the 8-point DFT $X[k], k = 0,1,2,3,4,5,6,7$ of $x[n]$.

d) (6) Carefully, **but approximately,** sketch $X[k], k = 0,1,2,3,4,5,6,7$ from your answer to part (c) above. In your sketch, you may ignore any linear phase factors.

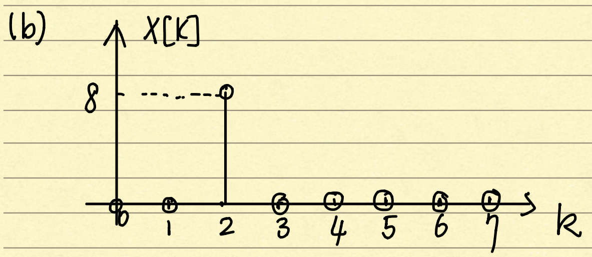
(a) $x[n] = e^{j2\pi(2)n/8}$

We know the DFT pairs: $e^{j2\pi k_0 n/N} \xrightarrow{DFT} N\delta[k-k_0]$ — 4pt

In our case $N=8$ $k_0=2$

$\therefore X[K] = 8\delta[k-2]$ — 1pt

(b)



— 2pt

(c) $e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi \, rep_{2\pi}[\delta(\omega-\omega_0)]$ — 2pt

$\omega_0 = \frac{2\times(5)\pi}{16} = \frac{5}{8}\pi$ → 2pt

$w[n] = u[n] - u[n-8]$

$W(\omega) = psinc_8(\omega)\, e^{-j\omega\frac{8-1}{2}} = psinc_8(\omega)\, e^{-j\omega\frac{7}{2}} = e^{-j\omega\frac{7}{2}} \frac{\sin(\frac{8}{2}\omega)}{\sin(\frac{1}{2}\omega)}$

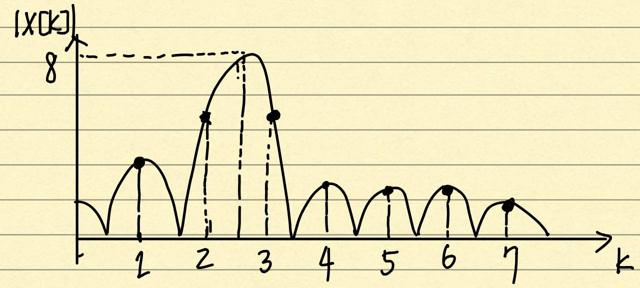
$= e^{-j\omega\frac{7}{2}} \frac{\sin(4\omega)}{\sin(\frac{1}{2}\omega)}$ — 2pt

$X_7(\omega) = W(\omega-\omega_0) = psinc_8(\omega - \frac{5}{8}\pi)\, e^{-j(\omega-\frac{5}{8}\pi)\cdot\frac{7}{2}}$ — 3pt

$\therefore X[K] = psinc_8(\frac{2\pi k}{8} - \frac{5}{8}\pi)\, e^{-j(\frac{2\pi k}{8} - \frac{5}{8}\pi)\cdot\frac{7}{2}}$

$= psinc_8(\frac{\pi k}{4} - \frac{5}{8}\pi)\, e^{-j(\frac{\pi k}{4} - \frac{5}{8}\pi)\cdot\frac{7}{2}}$ $k=0,1,2,\cdots,7$ — 3pt

(d)



6pt

3. (25 pts) Fast Fourier Transform Algorithm

a. (14) Derive a set of equations to show how a 10-point Discrete Fourier Transform (DFT) can be calculated in terms of two 5-point DFTs. (This is a 10-point Decimiation-in-Time FFT algorithm.

b. (11) Based on your answer to part a) above, draw a complete and fully labeled flow diagram for your 10-point FFT algorithm.

(a)

Based on the forward DFT, we have

$$\boxed{X^{(10)}[k] = \sum_{n=0}^{9} x[n]e^{-j2\pi\frac{kn}{10}} \quad \textbf{1 pt}}$$

Since we are formatting two 5-point DFTs, we decimate the input by 2.

Stage II: **2 pts**

Let $\boxed{n = 2m + p, m = 0, \dots, 4, p = 0, 1,}$ we can rewrite the DFT into:

$$\boxed{X^{(10)}[k] = \sum_{p=0}^{1}\sum_{m=0}^{4} x[2m+p]e^{-j2\pi\frac{k(2m+p)}{10}}} \quad \textbf{2 pts}$$

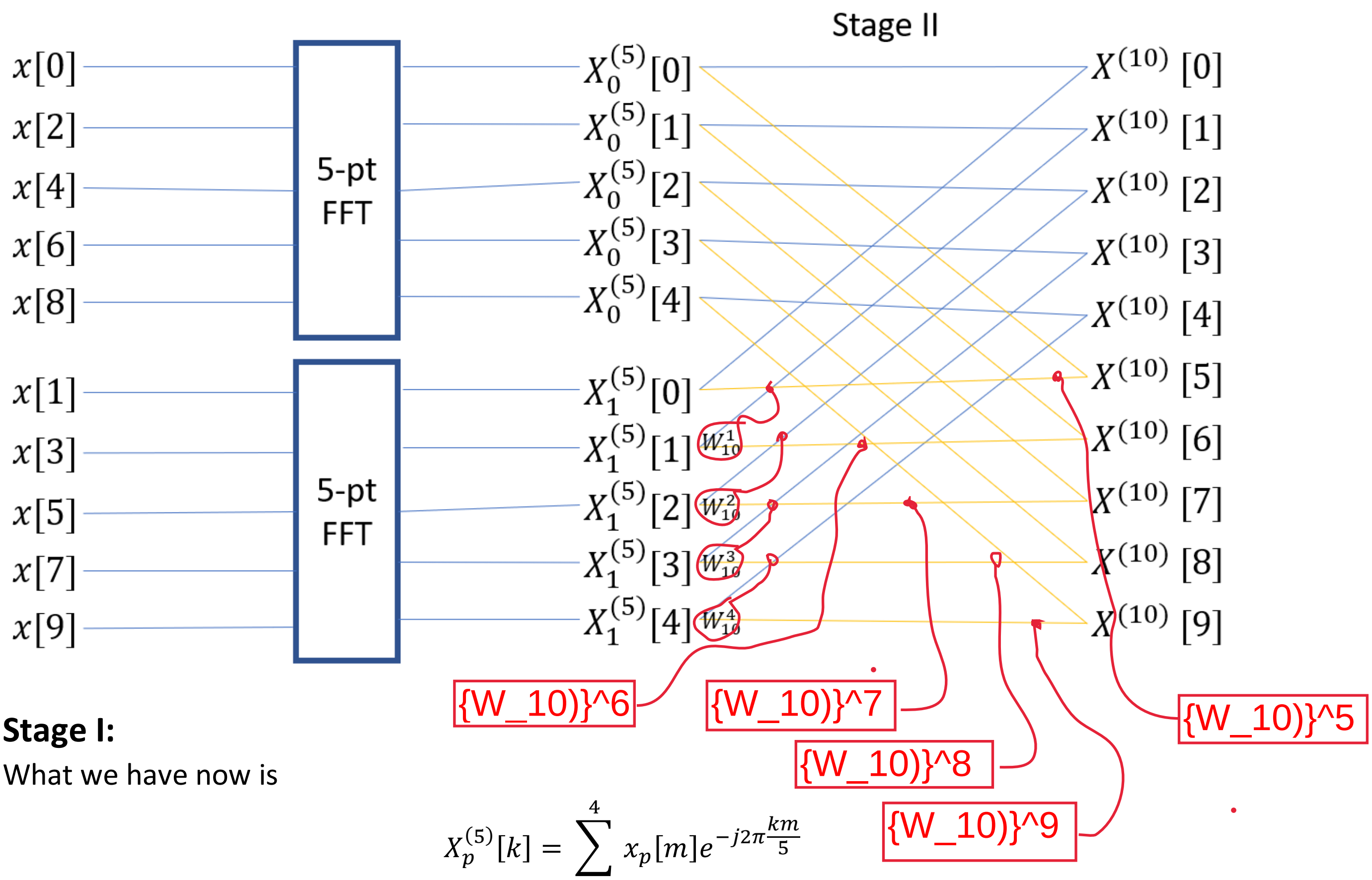
let $x_p[m] = x[2m+p], m = 0, \dots, 4, p = 0, 1$

$$\Rightarrow \boxed{X^{(10)}[k] = \sum_{p=0}^{1} e^{-j2\pi\frac{kp}{10}} \sum_{m=0}^{4} x_p[m]e^{-j2\pi\frac{km}{5}}} \quad \textbf{1 pt}$$

let $\boxed{X_p^{(5)}[k] = \sum_{m=0}^{4} x_p[m]e^{-j2\pi\frac{km}{5}}}$ **2 pts**

$$\Rightarrow \boxed{X^{(10)}[k] = \sum_{p=0}^{1} e^{-j2\pi\frac{kp}{10}} X_p^{(5)}[k] = \sum_{p=0}^{1} W_{10}^{kp} X_p^{(5)}[k]} \quad \textbf{3 pts}$$

where $W_N^k = e^{-j2\pi\frac{k}{N}}$



Stage I:

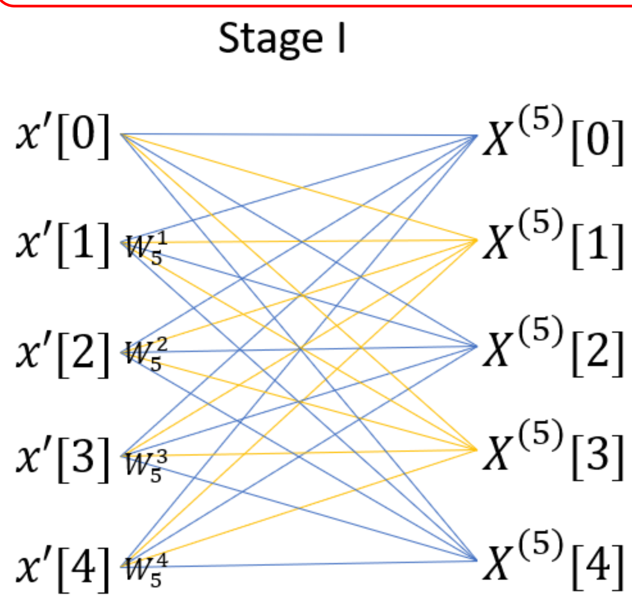
What we have now is

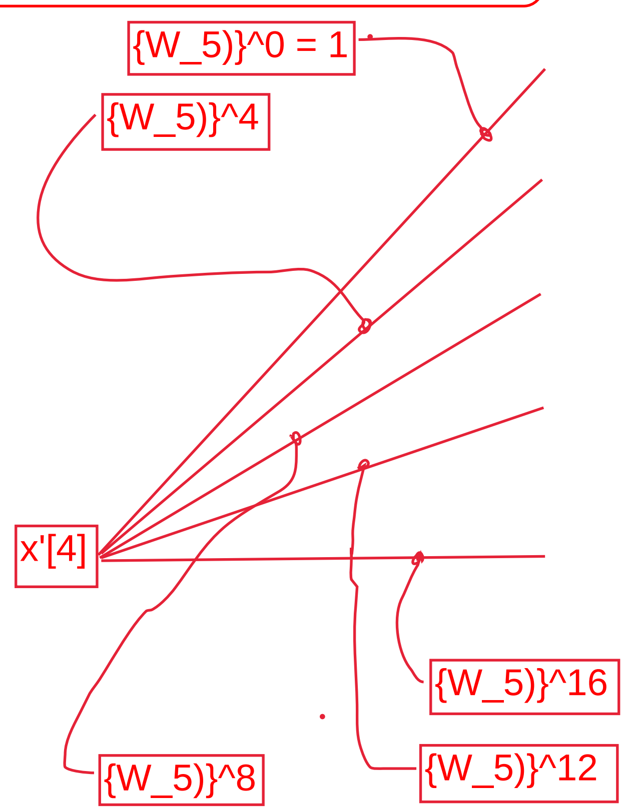
$$X_p^{(5)}[k] = \sum_{m=0}^{4} x_p[m] e^{-j2\pi\frac{km}{5}}$$

we can generalize it as

3 pts

$$\boxed{X^{(5)}[k] = \sum_{n=0}^{4} x'[n] e^{-j2\pi\frac{kn}{5}} = \sum_{n=0}^{4} x'[n] W_5^{kn} = x'[0] + x'[1]W_5^{k} + x'[2]W_5^{2k} + x'[3]W_5^{3k} + x'[4]W_5^{4k}}$$

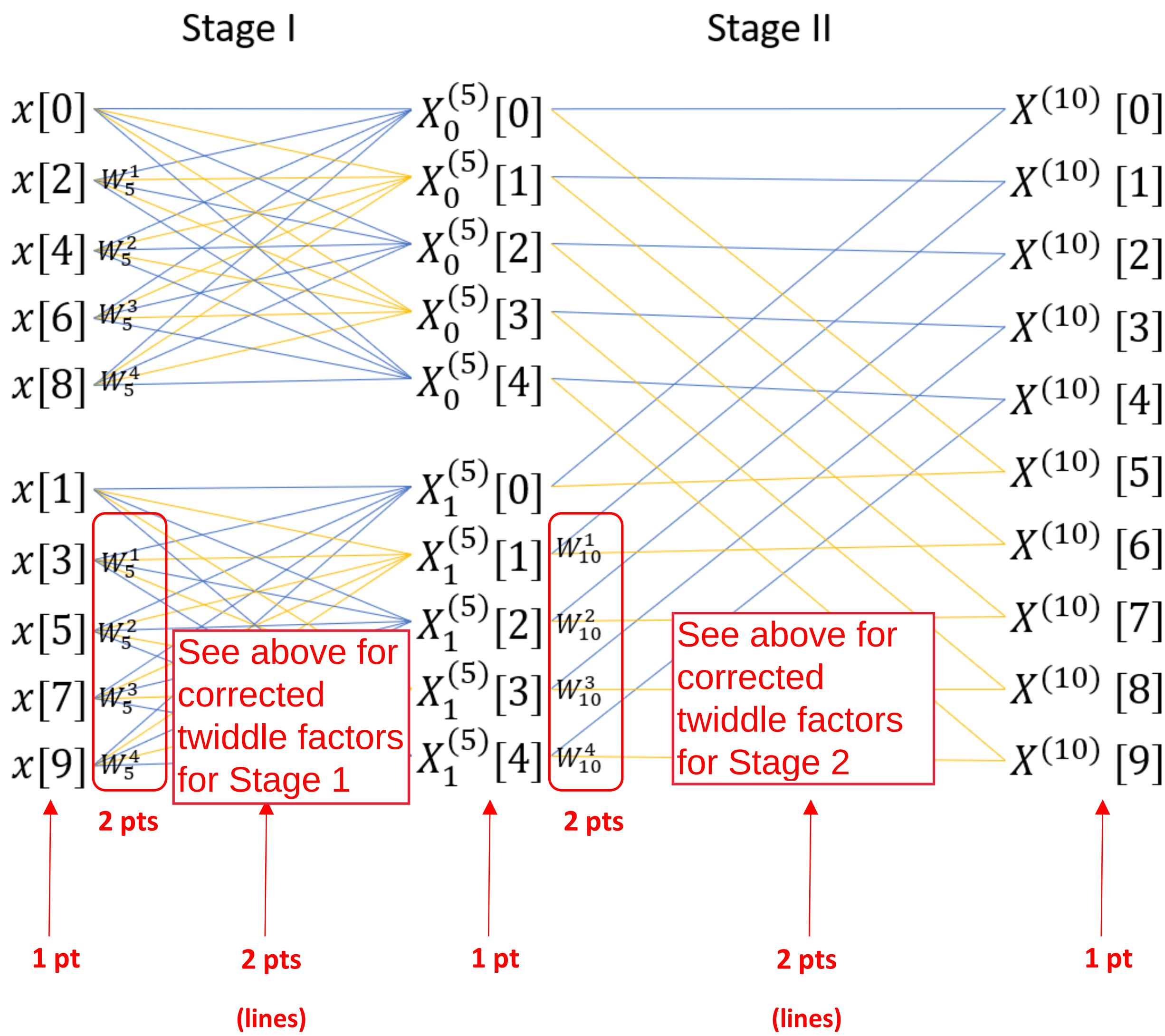




Note that I am only showing the corrected twiddle factors here for the x'[4] node. The other nodes would have similar twiddle factors. Everything here (i.e. the corrected twiddle factors) follows DIRECTLY from the above equations, which ARE correct.

(b)

The overall figure is



4. (25 pts) Consider a random variable X with density function

$$f_X(x)=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{else} \end{cases}.$$

a. (13) Find the mean and variance of X.

Suppose we generate a new random variable $Y=Q(X)$ by quantizing X according to the following quantizer:

$$Q(x)=\begin{cases} \frac{1}{4}, & 0\le x\le \frac{1}{2} \\ \frac{3}{4}, & \frac{1}{2}\le x\le 1 \end{cases}$$

b. (12) Provide an expression for the exact mean-squared quantization error $\varepsilon = E\left\{|Y-X|^2\right\}$ that only involves sums of integrals of powers of x. **NOTE:** You do not need to evaluate the integrals. This would require too much algebra.

a. $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}x^3\Big]_0^1 = \frac{2}{3}$ (5pt) $m_x = E[X] = 2/3$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)dx = \int_0^1 2x^3 dx = \frac{2}{4}x^4\Big]_0^1 = \frac{2}{4} = \frac{1}{2}$ (4pt) $\sigma_x^2 = E[x^2] - E[x]^2 = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$ (4pt)

b. $\varepsilon = E[|Y-X|^2] = \int_{-\infty}^{\infty} (Q(x)-x)^2 \cdot f_X(x)dx =$

$= \int_0^{1/2} (\frac{1}{4}-x)^2\, 2x\, dx + \int_{1/2}^1 (\frac{3}{4}-x)^2\, 2x\, dx =$

$= \int_0^{1/2} (\frac{1}{16} - \frac{1}{2}x + x^2)2x\, dx + \int_{1/2}^1 (\frac{9}{16} - \frac{3}{2}x + x^2)2x\, dx$ (12 pt)

Alternatively

$\varepsilon = E[(Y-X)^2] = E[Y^2 - 2XY + X^2] = E[Y^2] - 2E[XY] + E[X^2]$ (4pt)

$E[Y^2] = \int_0^{1/2} (1/4)^2 \cdot 2x\, dx + \int_{1/2}^1 (3/4)^2\, 2x\, dx = \frac{7}{16}$ (4pt)

$E[XY] = \int_0^{1/2} \frac{1}{4} 2x^2 dx + \int_{1/2}^1 \frac{3}{4} 2x^2 dx = \frac{11}{24}$ (4 pt)

$E[X^2] = \frac{1}{2}$