13 February 2019 **Name:____________________**

ECE 438 **Exam No. 1** **Spring 2019**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n+1] - 2x[n] + x[n-1].$$

a. (2) Is this system causal? Explain why or why not.

b. (2) Is this system bounded-input-bounded-output (BIBO) stable? Explain why or why not.

c. (11) Find the output $y[n]$ of this system when the input is

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{else} \end{cases}.$$

d. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.

Exam solution

Problem 1

6 pt

1. The system is not causal since the output needs future values x[n+1].

6 pt

2. The system is BIBO stable.
   Method (1).

   If $|x[n]| \leq M_x$, then,

   $|y[n]| = |x[n+1] - 2x[n] + x[n-1]| \leq |x[n+1] + 2x[n] + x[n-1]| \leq 4M_x$

   Method (2).

   $Y(\omega) = X(\omega)[e^{jw} - 2 + e^{-jw}]$
   $H(\omega) = (e^{jw} - 2 + e^{-jw}) = 2cos\omega - 2$
   So, the system is bounded-input-bounded-output stable.

7 pt

3. y[n] = x[n+1]-2x[n]+x[n-1]
   y[-1] = 1
   y[0] = -1
   y[1] = y[2] = y[3] = 0
   y[4] = -1
   y[5] = 1

   So,

$$y(n) = \begin{cases} -1, & n = 0, 4 \\ 1, & n = -1, 5 \\ 0, & else \end{cases}$$

6 pt

4. As showed in part (b),

$$H(\omega) = (e^{jw} - 2 + e^{-jw}) = 2\cos\omega - 2 = 2(\cos\omega - 1)$$

All correct / small typos. -- All points.
Minor mistakes. -- > 5-6 point
Some mistakes. --> 3-5 points
Major mistakes. --> 2-3 points
Do something but none of them is correct. --> 1 point
Do not do anything. ---> 0 point

2. (25 pts.) Consider a linear, time-invariant system with unit sample (impulse) response

$$h[n]=\left(\frac{1}{2}\right)^n u[n].$$

Find the response of this system $y[n]$ to the input

$$x[n]=\begin{cases} 1, & 10\le n\le 29 \\ 0, & \text{else} \end{cases},$$

by evaluating the convolution $y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

Your solution for $y[n]$ should be an analytical expression or expression(s) for the signal. It should not contain any summation signs $\sum$.

See the following page for an addendum to this solution.

2. (25 pts.) Consider a linear, time-invariant system with unit sample (impulse) response

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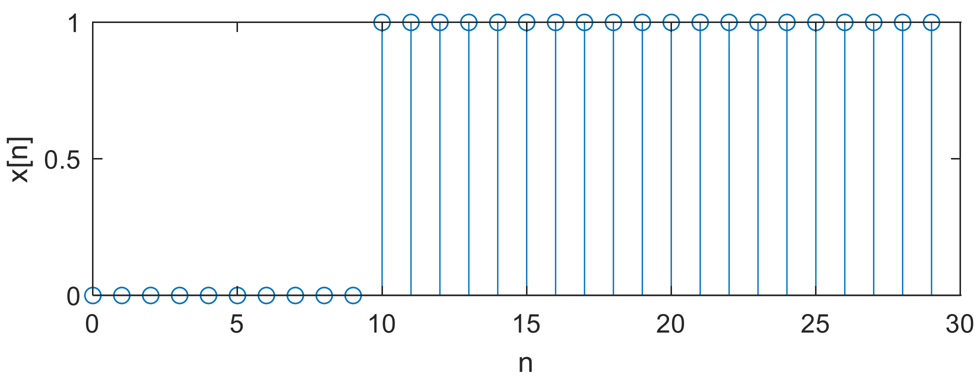
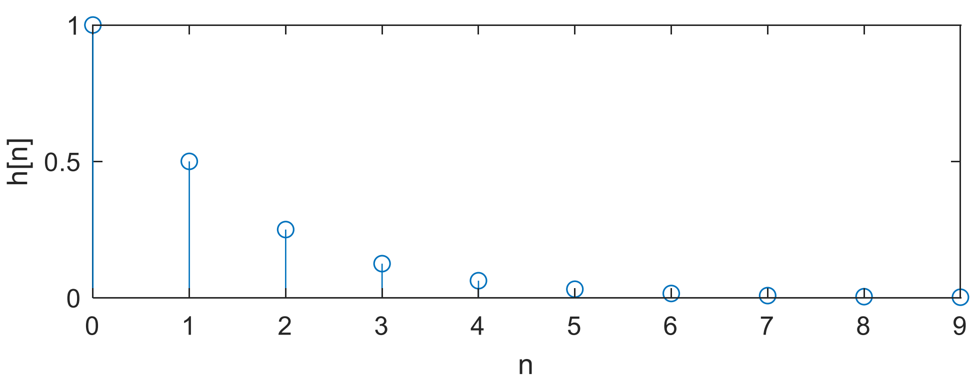




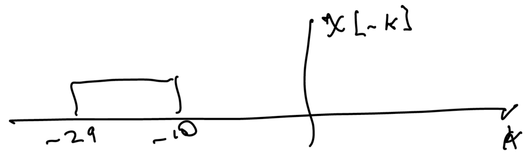
Addendum to Solution for Problem 2 on Exam 1 from Spring 2019

We are going to evaluate $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$.

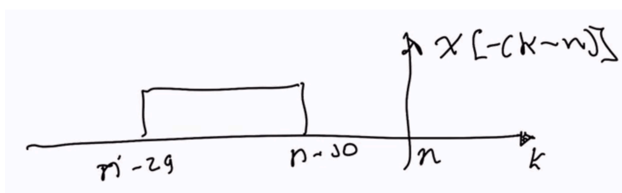
We have:

$$x[n] = \begin{cases} 1, & 10 \le n \le 29 \\ 0, & \text{else} \end{cases}$$

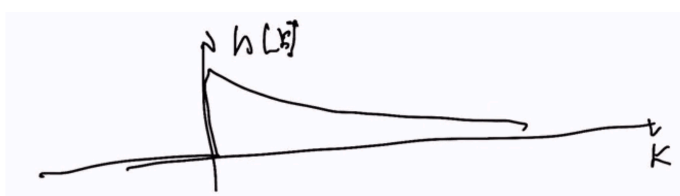
Then, let us consider $x[n-k] = x[-(k-n)]$ as a function of k for a fixed value of n.
First, we flip $x[n]$ to become $x[-k]$.



Then we shift it so that $x[0]$ occurs at $k = n$.



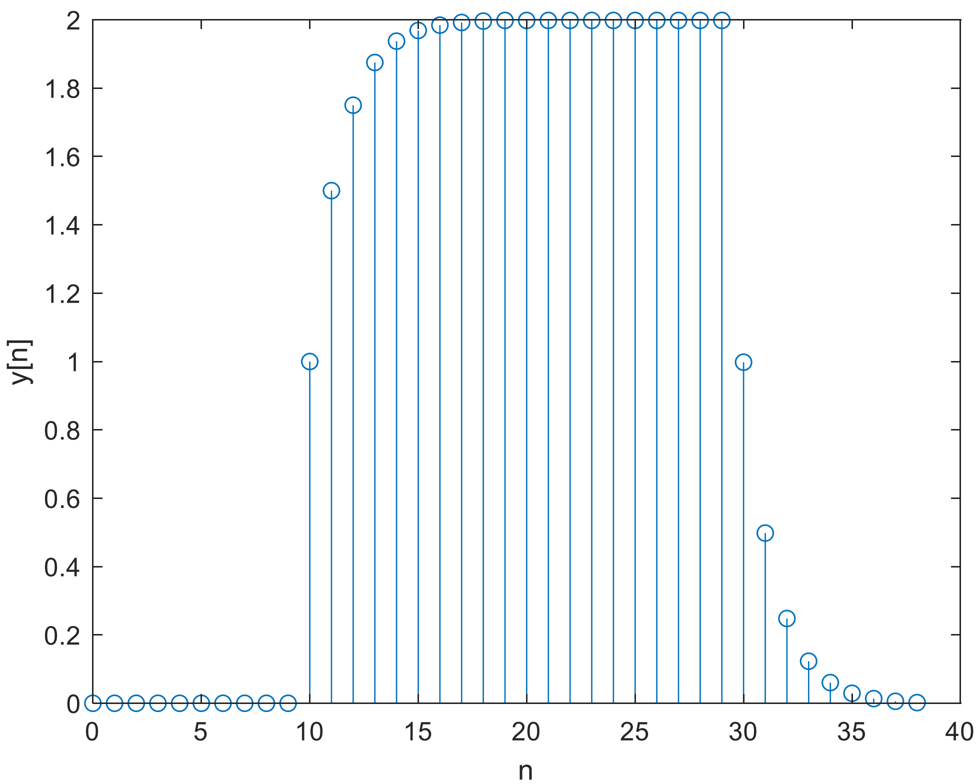
Now, we sketch $h[k]$



Case I: We can see that for $n-10<0$, $\Rightarrow n<10$ $x[n-k]$ and $h[k]$ will not overlap. So $y[n]=0$.

Case II: For $n-10 \ge 0$ and $n-29 \le 0$ $\Rightarrow n \le 29$, the two signals overlap for $0 \le k \le n-10$. So these are the limits of summation over k.

Case III: For $n-29>0$, the two signals only overlap for $n-29 \le k \le n-10$. So these are the limits of summation.



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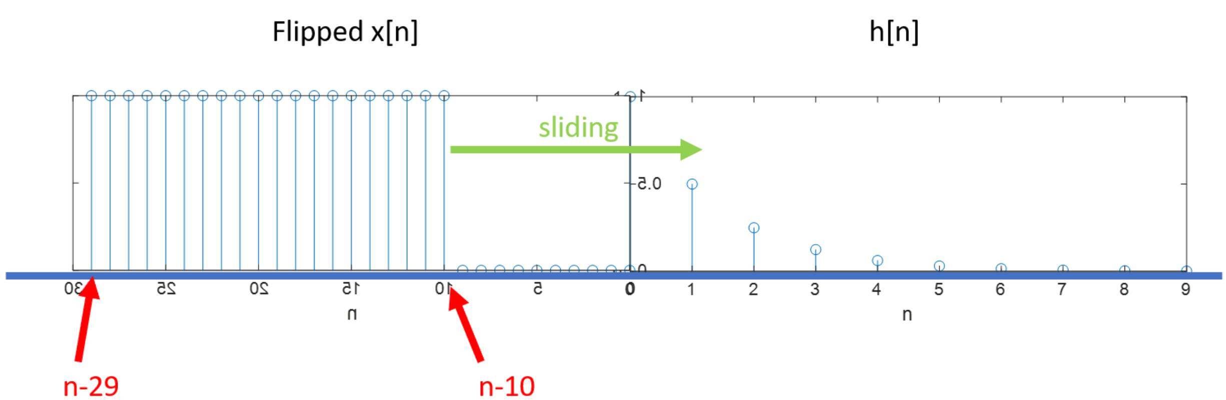


Case 1: $n \leq 9$ 3pt

$$y[n] = 0$$ 2pt

Case 2: $10 \leq n \leq 28$ 4pt

$$y[n] = \sum_{k=0}^{n-10} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{n-9}}{1 - \left(\frac{1}{2}\right)} = 2 - \left(\frac{1}{2}\right)^{n-10}$$

4pt 2pt

Case 3: $n \geq 29$

2pt

$$y[n] = \sum_{k=n-29}^{n-10} \left(\frac{1}{2}\right)^k$$

6pt

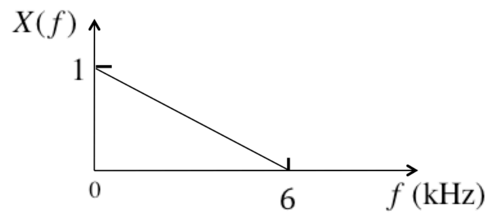
Let $l = k - n + 29$ which also means that $k = l + n - 29$, we can rewrite case 3 as

$$y[n] = \sum_{l=0}^{19} \left(\frac{1}{2}\right)^{l+n-29} = \left(\frac{1}{2}\right)^{n-29} \sum_{l=0}^{19} \left(\frac{1}{2}\right)^{l} = \left(\frac{1}{2}\right)^{n-29} \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)^{n-29} 2\left(1 - \left(\frac{1}{2}\right)^{20}\right)$$

$$= \left(\frac{1}{2}\right)^{n-30} \left(1 - \left(\frac{1}{2}\right)^{20}\right) \approx \left(\frac{1}{2}\right)^{n-30}$$

2pt

3. (25) Consider the real-valued continuous-time signal $x(t)$ with CTFT $X(f)$ shown below for positive frequencies only. Note that $X(-f)=X(f)$.



This signal is sampled at an 8 kHz rate with an ideal sampler (no prefilter; so it is not an ideal A/D) to generate the continuous-time (CT) sampled signal

$$x_s(t)=\sum_{k=-\infty}^{\infty} x(nT)\delta(t-nT),$$

where T is the sampling interval.

a. (6) Find a simple expression for the CTFT $X_s(f)$ of $x_s(t)$ in terms of a general $X(f)$. Your final answer should not contain any operators.

b. (6) Carefully sketch $X_s(f)$ for the specific CTFT $X(f)$ shown at the beginning of this problem statement. Be sure to dimension all important quantities on both the horizontal and vertical axes.

Suppose we wish to consider our sampled signal as a discrete-time (DT) signal defined according to $x[n]=x(nT)$.

c. (7) Find a simple expression for the DTFT $X_d(\omega)$ of $x[n]$ in terms of a general CTFT $X(f)$ of $x(t)$. Your final answer should not contain any operators.

d. (6) Carefully sketch $X_d(\omega)$ for the specific CTFT $X(f)$ shown at the beginning of this problem statement. Be sure to dimension all important quantities on both the horizontal and vertical axes.

1. $x_s(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT) = \text{comb}_T[x(t)]$

$$\text{comb}_T[x(t)] \xrightarrow{\text{CTFT}} \frac{1}{T}\,\text{rep}_{\frac{1}{T}}[X(f)]$$

$\therefore\ X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f - k\cdot\frac{1}{T})$ where $T = 1/8000$ sec

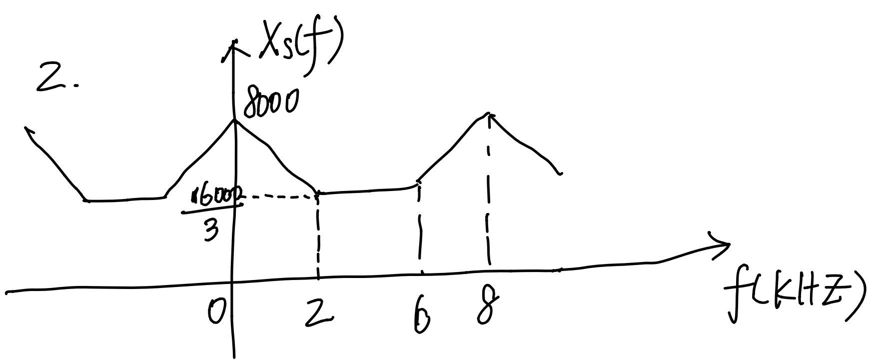
if students write

① $X_s(f) = \frac{1}{T}\,\text{rep}_{\frac{1}{T}}[X(f)]$ or

② $X_s(f) = \frac{1}{T}X(f) * \sum_{k=-\infty}^{+\infty} \delta(f - k\cdot\frac{1}{T})$

or any other forms involve operators, 3pt will be deducted

2.



If the answer has the same shape but minor mistake (magnitude, cut-off frequency, aliasing issue ---)

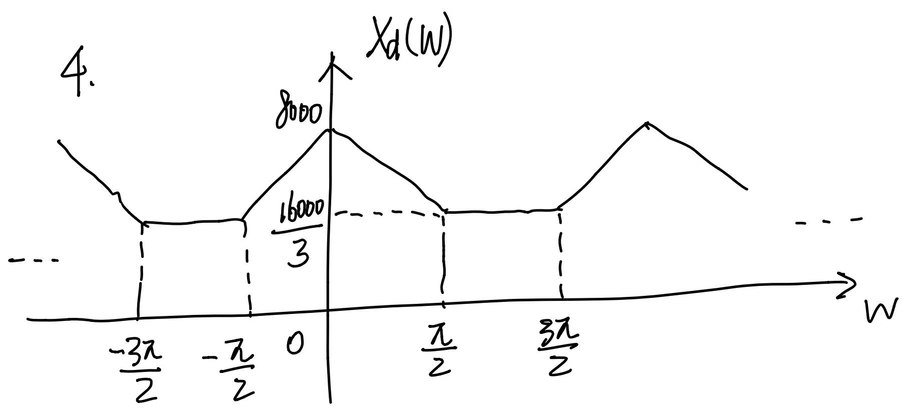
1pt off for each minor mistake

3. $X_d(W_d) = f_s \, rep_{f_s}[X_a(f_a)]_{f_a = \frac{W_d}{2\pi} \cdot f_s}$ where $f_s = 8000$

$$= 8000 \sum_{k=-\infty}^{+\infty} X\left(\frac{W}{2\pi} \cdot 8000 - 8000k\right) = 8000 \sum_{k=-\infty}^{+\infty} X\left(\frac{8000}{2\pi}(W - 2\pi k)\right)$$

if the answer involve any operators, 3 pt will be deducted.

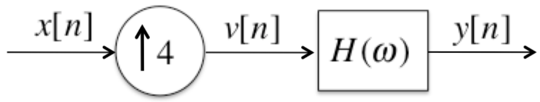
4.



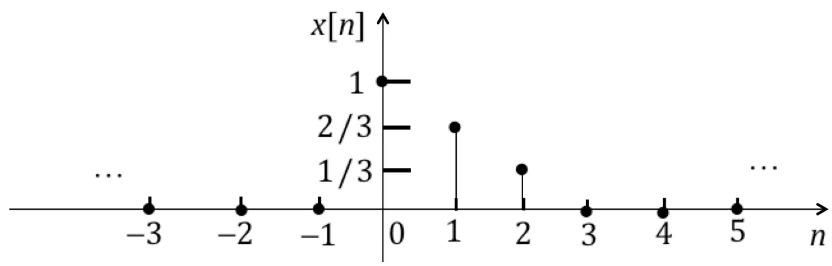
repeat every 2π

same grading rubric as part (b)

4. (25 pts) Consider the system shown below:

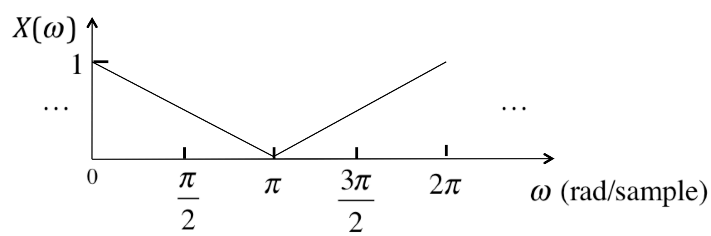


Assume that the signal $x[n]$ is given by



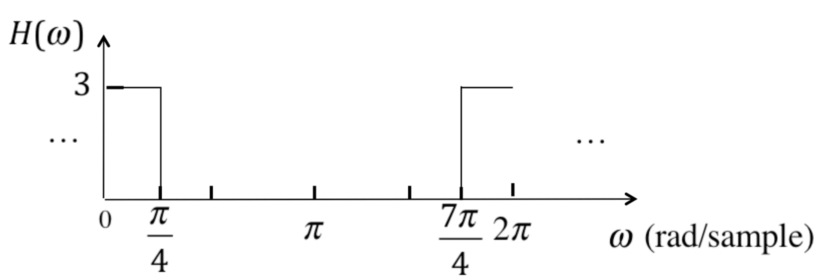
a) (5) Carefully sketch $v[n]$ for the specific signal $x[n]$ shown above. Be sure to dimension all important quantities.

Suppose that the DTFT $X(\omega)$ of the input $x[n]$ is given by:



b) (5) Carefully sketch the DTFT $V(\omega)$ of $v[n]$ for the specific DTFT $X(\omega)$ shown above. Be sure to dimension all important quantities.

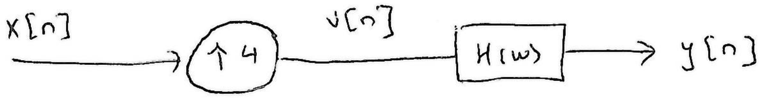
Suppose that the frequency response $H(\omega)$ of the filter is given by:



c) (5) Carefully sketch the DTFT $Y(\omega)$ of the output $y[n]$ for the given $V(\omega)$ from your answer to part b).

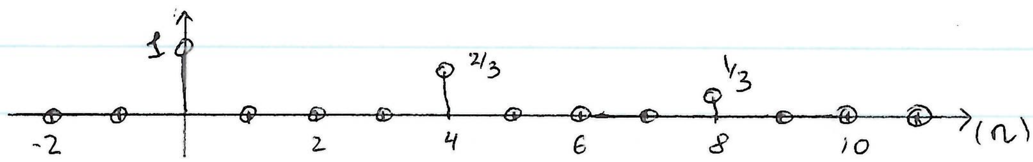
d) (5) The inverse DTFT of $H(\omega)$ is given by $h[n] = \text{sinc}(k/4)$. Carefully sketch $h[n]$. Be sure to dimension all important quantities.

e) (5) Using the given form of $x[n]$ shown at the beginning of this problem statement and your answer to part e) above, carefully sketch the output $y[n]$.



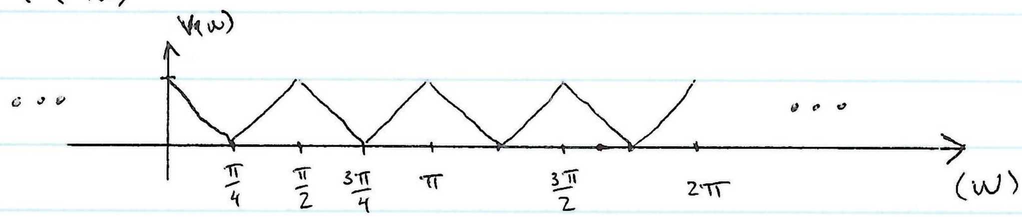
$x[n] = \{0, 0, \underset{\uparrow\, n=0}{1}, 2/3, 1/3, 0, 0\}$

a) $v[n] = \begin{cases} x[n/4] & , \; n/4 \text{ integer} \\ 0 & , \; \text{else} \end{cases}$



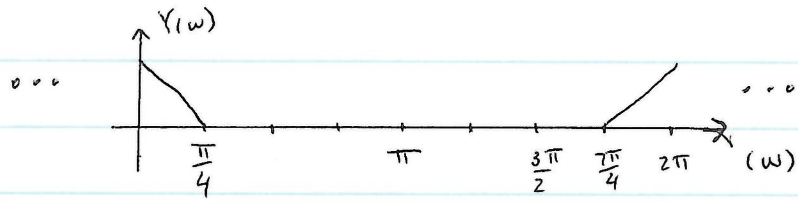
5 pts

b) $V(\omega) = X(4\omega)$



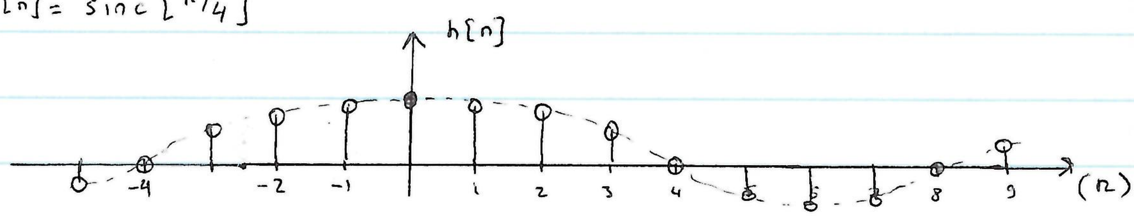
5 pts

c) $Y(\omega) = V(\omega)\, H(\omega)$



5 pts

d) $h[n] = \text{sinc}[n/4]$

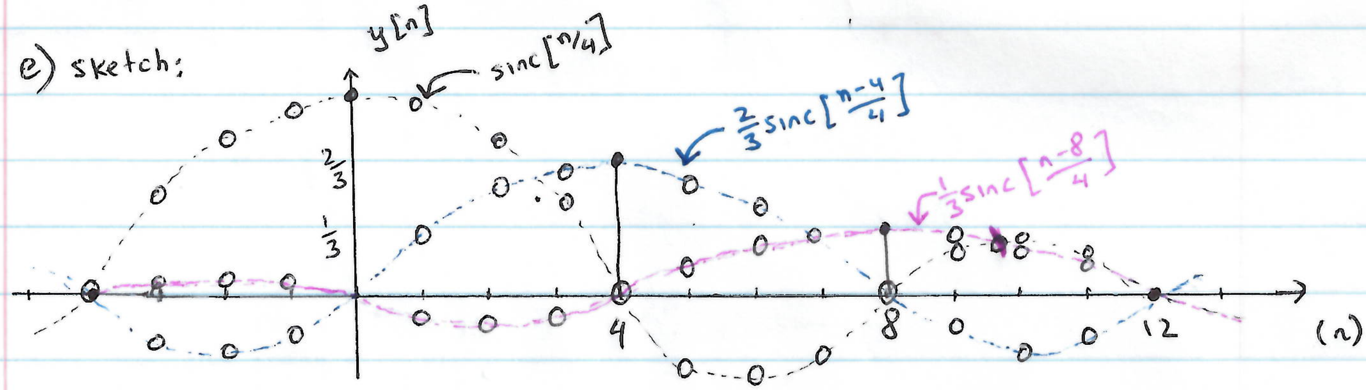


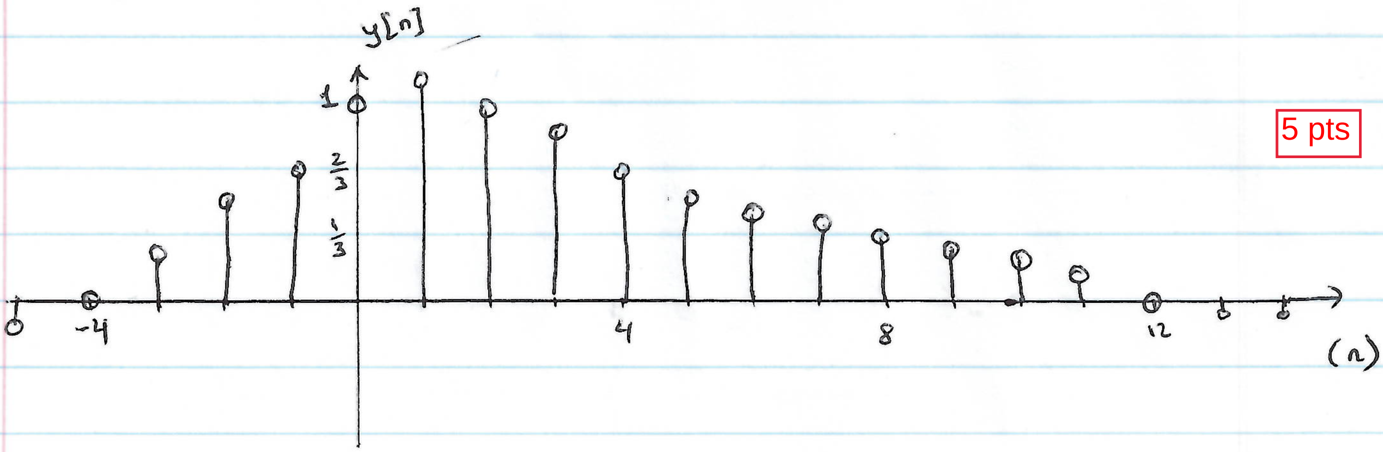
5 pts

e) $y[n] = v[n] * h[n] = \left(\delta[n] + \frac{2}{3}\delta[n-4] + \frac{1}{3}\delta[n-8]\right) * \text{sinc}(n/4) =$ ~~sinc(n/4) + sinc((n-4)/4) + sinc(~~

$= \text{sinc}\left[\frac{n}{4}\right] + \frac{2}{3}\text{sinc}\left[\frac{n-4}{4}\right] + \frac{1}{3}\text{sinc}\left[\frac{n-8}{4}\right]$

e) sketch:





5 pts

Additional plot generated in Matlab for larger range of n: