3 May 2018 **Name:_______________ Signature: _________________**

ECE 438 **Final Exam** **Spring 2018**

- You have 120 minutes to work the following five problems, which are worth 25 points each, for a total of 125 points.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem. Please note that the problems are arranged in the order that the topics were covered during the semester, not necessarily in the order of difficulty to solve them.

1. (25 pts.) Consider a system described by the following equation

$$y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right).$$

a. (9) Find the response $y[n]$ to the following input

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases}.$$

b. (9) Find a simple expression for the frequency response $H(\omega)$ for this system.

c. (7) From your answer to part (b), determine simple expressions for the magnitude and phase of the frequency response.

1. (continued – 1)

1. (continued – 2)

2. (25 pts.) Consider an N-point signal $x[n], n = 0,\ldots,N-1$, where N is assumed to be an even integer.

Define a new N-point signal $y[n] = \begin{cases} x[n], & n \text{ even} \\ -x[n], & n \text{ odd} \end{cases}$

a. (8) Express $y[n]$ as the product of $x[n]$ and a suitably chosen complex exponential signal that is a function of n.

b. (17) Find a simple expression for the N-point Discrete Fourier Transform (DFT) $Y[k], k = 0,\ldots,N-1$ of $y[n]$ in terms of the N-point DFT $X[k]$ of $x[n]$.

2. (continued – 1)

2. (continued – 2)

3. (25 pts.) Consider a random variable X with density function

$$f_X(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & \text{else} \end{cases} .$$

a. (3) Sketch $f_X(x)$.

b. (6) Find the mean and variance of X.

Suppose we generate a new random variable $Y = \begin{cases} X, & X \geq 0 \\ 0, & \text{else} \end{cases}$ where X is the random variable with density function $f_X(x)$ defined above.

b. (6) Find the mean and variance of Y.

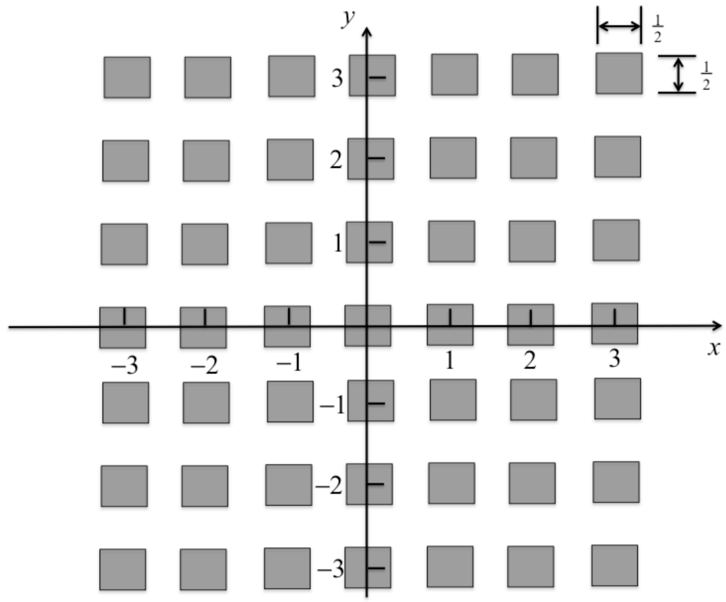
c. (6) Determine the correlation coefficient ρ_{XY} between X and Y.

d. (4) Find the density function $f_Y(y)$ for Y.

3. (continued – 1)

3. (continued – 2)

4. (25 pts) Consider the 2-D signal $f(x, y)$ shown below. This signal has value 1 in the shaded regions, and 0 elsewhere. It has finite extent, and is 0 outside the region shown. The units of x and y are inches.



a. (8) Find a simple expression for $f(x, y)$ using standard functions and operators that were defined in class.

b. (11) Find a simple expression for the CSFT $F(u,v)$ of $f(x, y)$. Your answer should not contain any operators. Any summations that can be evaluated in closed-form should be replaced by their closed-form equivalents.

c. (6) Sketch what the spectrum $F(u,v)$ would look like. Be sure to dimension all important quantities, including units for the dimensions. Your sketch should indicate that you clearly understand what the spectrum looks like.

4. (continued – 1)

4. (continued – 2)

5. (25 pts) Consider a spatial filter with *non-separable* point spread function $h[m,n]$ given below

	$h[m,n]$	$n=-1$	0	1
m	-1	0	-1	0
	0	1	1	1
	1	0	-1	0

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 11×11 set of pixels in the input image.

0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, and the $\mu = \nu$ axis.

c. (3) Using your results from parts a) and b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

5. (continued – 1)

5. (continued – 2)

1. (25)	_______
2. (25)	_______
3. (25)	_______
4. (25)	_______
5. (25)	_______
Total (125)	_______

ECE 438 Final Exam Solution -- Spring 2018

Problem 1

a) $y[n] = \begin{cases} 0, & n<0 \\ 1/5, & n=0 \\ 2/5, & n=1 \\ 3/5, & n=2 \\ 4/5, & n=3 \\ 1, & n\geq 4 \end{cases}$ (9 pts)

b) $h[n] = \frac{1}{5}(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4])$.

Ist method: (7 pts)

$$H(\omega) = \frac{1}{5}(1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}) =$$

$$= \frac{1}{5}\sum_{k=0}^{4} e^{-j\omega k} = \frac{1}{5}\cdot\frac{1-e^{-j\omega 5}}{1-e^{-j\omega 4}} = \frac{1}{5}\, \frac{e^{-j\omega\frac{5}{2}}\left(e^{j\omega\frac{5}{2}} - e^{-j\omega\frac{5}{2}}\right)}{e^{-j\omega\frac{1}{2}}\left(e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}\right)}$$

$$= \frac{1}{5} e^{-j\omega\left(\frac{5}{2}-\frac{1}{2}\right)} \cdot \frac{\cancel{2j}\sin\left(\frac{5}{2}\omega\right)}{\cancel{2j}\sin\left(\frac{1}{2}\omega\right)} =$$

$$= \frac{1}{5} e^{-j\omega\frac{4}{2}} \cdot \frac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = \frac{1}{5}\, \mathrm{psinc}_5(\omega)\cdot e^{-j\omega 2}.$$

(2 pts for simplifying)

IInd method: (7 pts)

$$H(\omega) = \frac{1}{5}\left(1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}\right) =$$

$$= \frac{1}{5}\cdot e^{-j2\omega}\cdot\left(e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}\right) =$$

$$= \frac{1}{5} e^{-j2\omega}\cdot\left(2\cos(2\omega) + 2\cos\omega + 1\right).$$ (2 pts)

c) Based on the answers in part b:

Ist method:

(4pts) $|H(\omega)| = \left|\frac{1}{5} e^{-j\omega 2} \cdot psinc_5(\omega)\right| = \left|\frac{1}{5} psinc_5(\omega)\right|$, simple expression,

(3pts) $\angle H(\omega) = \begin{cases} -2\omega, & psinc_5(\omega) \geq 0 \\ -2\omega \pm \pi, & psinc_5(\omega) < 0 \end{cases}$

IInd method:

(4pts) $|H(\omega)| = \left|\frac{1}{5} e^{-j\omega 2} \cdot (2\cos(2\omega) + 2\cos(\omega) + 1)\right| =$

$= \left|\frac{1}{5}(2\cos(2\omega) + 2\cos(\omega) + 1)\right|$

(3pts) $\angle H(\omega) = \begin{cases} -2\omega, & 2\cos(2\omega) + 2\cos(\omega) + 1 \geq 0 \\ -2\omega \pm \pi, & 2\cos(2\omega) + 2\cos(\omega) + 1 < 0 \end{cases}$.

ECE 438 Final exam solution

2. a) **8pts**

$y[n] = x[n]e^{j\pi n}$

or $y[n] = x[n]e^{-j\pi n}$ **8 pts**

b) **17pts**

Since $x[n]e^{j2\pi n k_0/N} \xleftrightarrow{DFT} X[k-k_0]$ **10 pts**

here, $k_0 = N/2$ **3 pts**

so. $x[n]e^{j2\pi n(N/2)/N} = x[n]e^{j\pi n} \xleftrightarrow{DFT} X[k-\frac{N}{2}]$

Similarly,

$x[n]e^{-j\pi n} \xleftrightarrow{DFT} X[k+\frac{N}{2}]$ **3pts**

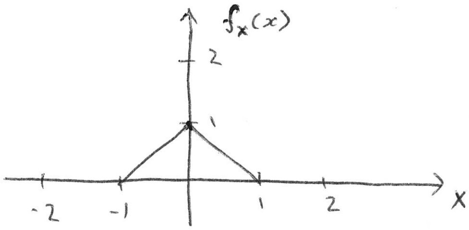
$$Y[k] = \begin{cases} X[k-\frac{N}{2}] & k \geq \frac{N}{2} \\ X[k+\frac{N}{2}] & k < \frac{N}{2} \end{cases}$$

N is even. **1 pt**

3. (continued – 2)

$$f_X(x) = \begin{cases} 1-|x| & , |x|<1 \\ 0 & , \text{else} \end{cases} = \begin{cases} 1+x & , -1 \le x \le 0 \\ 1-x & , 0 < x \le 1 \\ 0 & , \text{else} \end{cases}$$

a.



3 pts

b. $E[X] = \int x f_X(x)dx = \int_{-1}^{0} x\cdot(1+x)dx + \int_0^1 x(1-x)dx = \left.\frac{x^2}{2}+\frac{x^3}{3}\right]_{-1}^{0} + \left.\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = -\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) = 0$

$E[X^2] = \int x^2 f_X(x)dx = \int_{-1}^{0} x^2(1+x)dx + \int_0^1 x^2(1-x)dx = \left.\frac{x^3}{3}+\frac{x^4}{4}\right]_{-1}^{0} + \left.\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = -\left(-\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{6}$

$\sigma_X^2 = E[X^2]-E[X]^2 = \frac{1}{6}$

6 pts

c. $Y = g(X) = \begin{cases} X & , X \ge 0 \\ 0 & , \text{else} \end{cases}$

$E[Y] = \int g(x)\cdot f_X(x)dx = \int_{-\infty}^{0} 0\cdot f_X(x)dx + \int_0^{\infty} x\cdot f_X(x)dx = \int_0^1 x(1-x)dx = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$

$E[Y^2] = \int g^2(x) f_X(x)dx = \int_0^{\infty} x^2 f_X(x)dx = \int_0^1 x^2(1-x)dx = \frac{1}{3}-\frac{1}{4} = \frac{1}{12}$

6 pts

$\sigma_Y^2 = E[Y^2]-E[Y]^2 = \frac{1}{12}-\frac{1}{36} = \frac{1}{18}$

d. $\rho_{XY} = \dfrac{E[XY]-E[X]E[Y]}{\sigma_X\ \sigma_Y}$

$E[XY] = \int x\, g(x)\, f_X(x)dx = \int_0^1 x\cdot x\cdot(1-x)\,dx = \frac{1}{12}$

$\rho_{XY} = \dfrac{\frac{1}{12} - 0\cdot\frac{1}{6}}{\sqrt{\frac{1}{6}}\cdot\sqrt{\frac{1}{18}}} = \frac{1}{12}\cdot 6\sqrt{3} = \frac{\sqrt{3}}{2}$

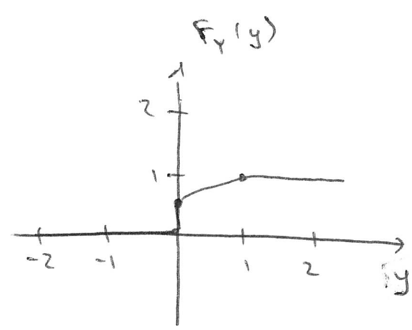
6 pts

$F_Y(y) =$

i. $y<0$ $F_Y(y)=0$

ii. $0 \le y < 1$ $F_Y(y) = \int_{-1}^{0} 1+x\,dx + \int_0^y (1-x)dx = \left.x+\frac{x^2}{2}\right]_{-1}^{0} + \left.x-\frac{x^2}{2}\right]_0^y = \frac{1}{2}+y-\frac{y^2}{2}$

iii. $y>1$ $F_Y(y) = \int_{-1}^{1} f_X(x)dx = \int_{-1}^{0} 1+x\,dx + \int_0^1 1-x\,dx = \frac{1}{2}+\frac{1}{2} = 1$



$f_Y(y) = \begin{cases} \frac{1}{2}\delta(y) & , y=0 \\ 1-y & , 0<y\le 1 \\ 0 & , y\ \text{else} \end{cases}$

4 pts

4. (continued – 1)

8pts (a) $f(x, y) = \underbrace{\text{rep}_{1,1}\{\text{rect}(2x, 2y)\}}_{(5)} \cdot \underbrace{\text{rect}\left(\frac{x}{7}, \frac{y}{7}\right)}_{(3)}$

- Used correct functions and operators (4)
- (5) pts given if the left term is correct, (3) if the right term is correct.
- (2)pts off for each wrong constant.

11 pts (b) $F(u, v) = \text{comb}_{1,1}\left\{\frac{1}{4}\,\text{sinc}\left(\frac{u}{2}, \frac{v}{2}\right)\right\} ** 49\,\text{sinc}(7u, 7v)$ – (6)

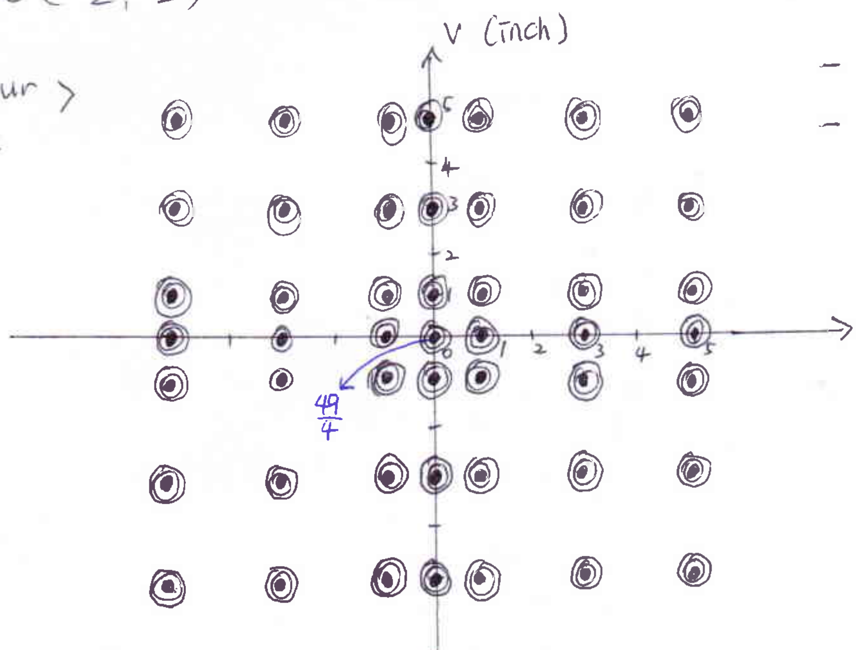
$$= \frac{49}{4}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{m}{2}, \frac{n}{2}\right)\delta(u-m, v-m)$$

$$** \text{sinc}(7u, 7v) \quad – (2)$$

$$= \frac{49}{4}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{m}{2}, \frac{n}{2}\right)\text{sinc}\left(7(u-m), 7(v-n)\right) \quad – (3)$$

6 pts (c) Note that $\text{sinc}\left(\frac{m}{2}, \frac{n}{2}\right) = 0$ if m or n is an even number.

<Contour> Plot



- • : top tip of each sinc
- Magnitude gets smaller as sinc gets farther from the origin.

- Shape of the spectrum (3)
- Quantities (1)
- axis label (1) and axis unit (1)

Question 1 5

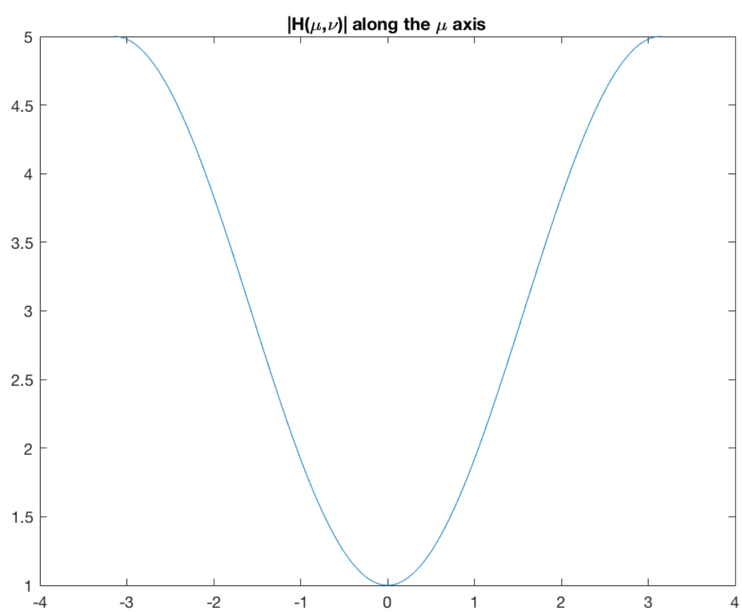
a. 10 pts

0	-1	-1	-1	-1	-1	0	0	0	0	0
1	1	2	2	2	1	0	0	0	0	0
1	0	1	1	1	1	1	0	0	0	0
1	0	1	1	1	1	1	1	0	0	0
1	0	1	1	1	1	1	1	1	0	0
1	0	1	1	1	1	1	1	1	2	1
1	0	1	1	1	1	1	1	1	0	0
1	0	1	1	1	1	1	1	0	0	0
1	0	1	1	1	1	1	0	0	0	0
1	1	2	2	2	1	0	0	0	0	0
0	-1	-1	-1	-1	-1	0	0	0	0	0

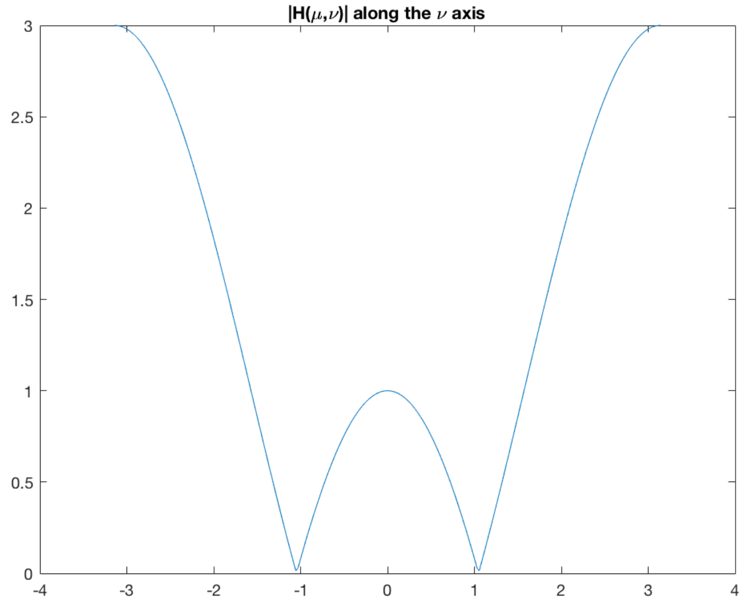
b.

$$\begin{aligned} h[m,n] &= -\delta[m+1,n] + \delta[m,n+1] + \delta[m,n] + \delta[m,n-1] - \delta[m-1,n] \\ \therefore H(\mu,\nu) &= -e^{j\mu} + e^{j\nu} + 1 + e^{-j\nu} - e^{-j\mu} \\ &= -2\cos\mu + 2\cos\nu + 1 \end{aligned}$$

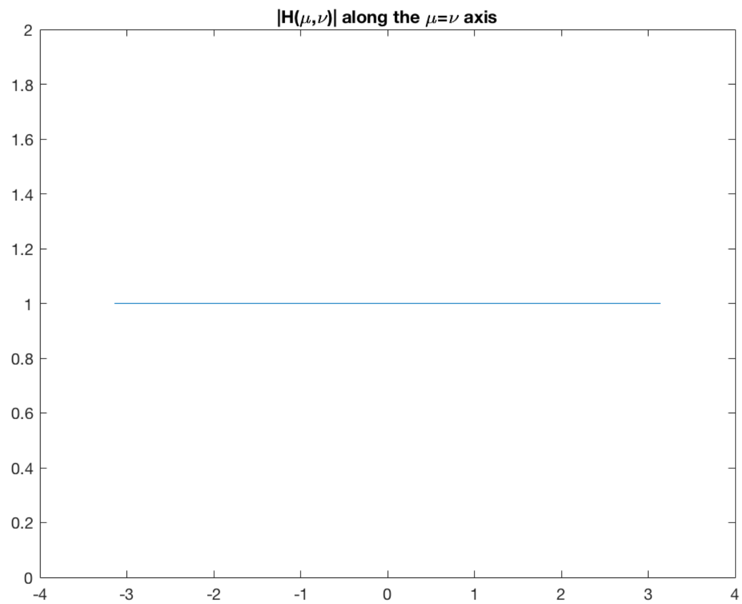
3 pts



3 pts



3 pts



3 pts

c. 3 pts

Along vertical direction, the high frequency component is enhanced. The edge is emphasized.

Along horizontal direction, the high frequency component is also enhanced, but the component at $\frac{\pi}{3}$ is suppressed.
The filter does not affect the diagonal direction.
Since the DC response is 1, the center region of 1s is maintained.