18 April 2018 **Name:____________________**

ECE 438 **Exam No. 3** **Spring 2018**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a random variable X with density function

$$f_X(x)=\begin{cases} 2x, & 0\le x\le 1 \\ 0, & \text{else} \end{cases} .$$

a. (9) Find the mean and variance of X .

Suppose we generate a new random variable $Y=Q(X)$ by quantizing X according to the following quantizer:

$$Q(x)=\begin{cases} 0, & 0\le x<\frac{1}{2} \\ 1, & \frac{1}{2}\le x\le 1 \end{cases}$$

b. (8) Find the mean and variance of Y .

c. (8) Calculate the exact mean-squared quantization error $\phi=E\left\{\left(Y-X\right)^2\right\}$.

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c. (8) Calculate the exact mean-squared quantization error $\phi=E\left\{(Y-X)^2\right\}$.

a. $E[X]=\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot 2x\cdot dx = \frac{2}{3}x^3\Big]_0^1 = \frac{2}{3}$

$E[x^2]=\int_0^1 x^2\cdot 2x\,dx = \frac{2}{4}x^4\Big]_0^1 = \frac{2}{4}=\frac{1}{2}$

$\text{Var}(X)= E[x^2]-E[x]^2 = \frac{1}{2}-\left(\frac{2}{3}\right)^2 = \frac{1}{18}$

$\text{mean}(x) = E[x] = \frac{2}{3}$

1. (continued - 1)

b. $E[Y] = \int_{-\infty}^{\infty} Q(x)\, f_X(x)\, dx = \cancelto{0}{\int_{0}^{1/2} 0 \cdot 2x\, dx} + \int_{1/2}^{1} 1 \cdot 2x\, dx =$

$$= x^2 \Big]_{1/2}^{1} = 1 - 1/4 = \frac{3}{4}$$

$E[Y^2] = \int_{-\infty}^{\infty} Q^2(x)\, f_X(x)\, dx = \cancelto{0}{\int_{0}^{1/2} 0 \cdot 2x \cdot dx} + \int_{1/2}^{1} 1 \cdot 2x \cdot dx = \frac{3}{4}$

$\mathrm{var}(Y) = E[Y^2] - E[Y]^2 = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$

$\mathrm{mean}(Y) = E[Y] = \frac{3}{4}$

c. $\phi = E[(Y-X)^2] = E[Y^2 - 2XY + X^2] = E[Y^2] - 2E[XY] + E[X^2]$

$E[XY] = \int_{-\infty}^{\infty} x \cdot Q(x)\, f_X(x)\, dx = \cancelto{0}{\int_{0}^{1/2} x \cdot 0 \cdot 2x\, dx} + \int_{1/2}^{1} x \cdot 1 \cdot 2x\, dx =$

$$= \frac{2}{3} x^3 \Big]_{1/2}^{1} = \frac{2}{3}\left(1 - \frac{1}{8}\right) = \frac{7}{12}$$

$\phi = E[Y^2] + E[X^2] - 2 \cdot E[XY] = \frac{3}{4} + \frac{1}{2} - 2 \cdot \frac{7}{12} = \frac{9+6-14}{12} = \frac{1}{12}$

2. (25) Let $X[n]$ be a sequence of identically distributed, independent random variables with mean zero and variance unity.

   Suppose that this sequence is processed with the following filter to generate the output sequence $Y[n]$:

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

   a. (3) Find the mean of the output sequence $Y[n]$.

   b. (12) Find the cross-correlation $r_{XY}[n]$ between X and Y.

   c. (10) Find the autocorrelation $r_{YY}[n]$ of the output Y.

ECE438 Exam3 Solution

2 (a)

$$Y[n] = \frac{1}{2}Y[n-1] + x[n]$$

$$Y[n] = \sum_k h[n-k]\,x[k]$$

$$E[Y[n]] = E\left\{\sum_k h[n-k]x[k]\right\}$$

$$= \sum_k h[n-k]E[x[k]] = 0 \qquad \boxed{5 \text{ pts}}$$

(b) $r_{xx}(n_1, n_2) = E[x[n_1]x[n_2]]$

because x[n] is independent and identically distributed.

$$r_{xx}(n_1, n_2) = \begin{cases} E[x[n_1]]\cdot E[x[n_2]] & n_1 \neq n_2 \\ E[x^2[n_1]] & n_1 = n_2 \end{cases}$$

$$= \begin{cases} 0 & , n_1 \neq n_2 \\ 1 & , n_1 = n_2 \end{cases}$$

$$\therefore \quad r_{xx}[n] = \delta[n] \qquad \boxed{1 \text{ pts}}$$

$$r_{xy}[n] = h[n] * r_{xx}[n] \qquad \boxed{5 \text{ pts}}$$

$$r_{xy}[n] = h[n] * \delta[n]$$

$$= h[n] = \left(\frac{1}{2}\right)^n u[n] \quad , \quad \text{RoC} : |z| > \frac{1}{2} \qquad \boxed{4 \text{ pts}}$$

includes the unit circle.

See following page for additional detail on how h[n] is obtained

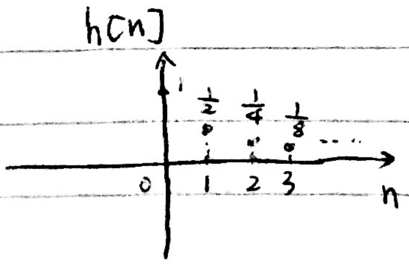
(c) $r_{yy}[n] = h[n] * r_{xy}[n] = h[n] * h[-n] \qquad \boxed{5 \text{ pts}}$

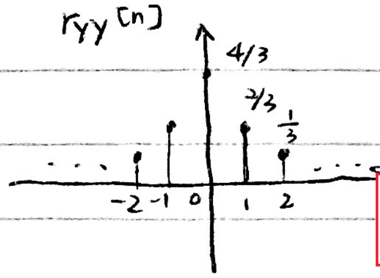
$$= \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^{-n} u[-n]$$

n	0	±1	±2	±3	....	±k
$r_{yy}[n]$	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$		$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n+k}$

$\boxed{5 \text{ pts}}$

Note that meaning of indices n and k are switched here.





See following page for more formal derivation of r_YY[n]

Derivation of impulse response $h[n]$ for filter

Filter equation: $y[n]=\frac{1}{2}y[n-2]+x[n]$

$\Rightarrow Y(z)=\frac{1}{2}Y(z)z^{-1}+X(z)$.

Rearranging, we get

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-\frac{1}{2}z^{-1}}.$$

We assume a casual filter; so $h[n]=0, n<0$, and $ROC_{H(z)}=\left\{z:|z|>\frac{1}{2}\right\}$

We can find inverse transform from the Formula sheet as $h[n]=\left(\frac{1}{2}\right)^n u[n]$.

Computation of $r_{YY}[n]$

We can note from class that if a linear, time-invariant filter has an input that is wide-sense stationary, then the output will be wide-sense stationary; and $r_{YY}[n]=h[n]*r_{XY}[-n]$. That is we time-reverse $r_{XY}[n]$, and then filter it.

From the above, we have that

$$\begin{aligned}
r_{YY}[n]&=h[n]*h[-n]\\
&=\sum_{k=-\infty}^{\infty}h[n-k]h[-k]\\
&=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\right)^{n-k}u[n-k]\left(\frac{1}{2}\right)^{-k}u[-k]\\
&=\sum_{k=-\infty}^{\min(0,n)}\left(\frac{1}{2}\right)^{n-k}\left(\frac{1}{2}\right)^{-k}\\
&=\left(\frac{1}{2}\right)^{n}\sum_{k=-\infty}^{\min(0,n)}\left(\frac{1}{4}\right)^{-k}
\end{aligned}$$

Let's first consider the case where $n \geq 0$. Then we have

$$\begin{aligned} r_{YY}[n] &= \left(\frac{1}{2}\right)^{n} \sum_{k=-\infty}^{0} \left(\frac{1}{4}\right)^{-k} \\ &= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} \\ &= \left(\frac{1}{2}\right)^{n} \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3}\left(\frac{1}{2}\right)^{n} \end{aligned}$$

But, we know that $r_{YY}[n] = r_{YY}[-n]$. So we obtain

$$r_{YY}[n] = \frac{4}{3}\left(\frac{1}{2}\right)^{|n|}$$

We can also get the result for $n < 0$ by solving

$$r_{YY}[n] = \left(\frac{1}{2}\right)^{n} \sum_{k=-\infty}^{n} \left(\frac{1}{4}\right)^{-k}.$$

But it is a lot more work.

3. (25) Consider the discrete-time signal

$$s[n] = \begin{cases} \cos(\pi n/10), & n < 0 \\ \cos(\pi n/2), & n \geq 0 \end{cases} .$$

Define the short-time discrete-time Fourier transform (STDTFT) as

$$\tilde{S}(\omega,n) = \sum_{k=-\infty}^{\infty} s[k]w[n-k]e^{-jwk} ,$$

where the window function is given by

$$w[n] = \begin{cases} 1, & |n| < 10 \\ 0, & \text{else} \end{cases} .$$

a. (18) Find simple closed form expressions for $\tilde{S}(\omega,n)$ for the three values of $n = -20, 0, 20$.

b. (7) Sketch $|\tilde{S}(\omega,n)|$ for $0 \leq \omega \leq \pi$ and for all n. Be sure to label and dimension all important quantities.

Prob. 3.

a. (18) Find simple closed form expressions for $\tilde{S}(w,n)$ for the three values of $n = -20, 0, 20$.

(i) If $n < -10$,

$$x[n] = cos\left(\frac{\pi n}{10}\right)$$

$$\tilde{S}(w,n) = \sum_k cos\left(\frac{\pi k}{10}\right) w[n-k]e^{-jwk}$$

The following DTFT property and pair are used.

$$x[n]y[n] < \frac{DTFT}{\quad} > \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w-\mu)Y(\mu)d\mu$$

$$cos\left(\frac{\pi n}{10}\right) < \frac{DTFT}{\quad} > \pi\left(\delta\left(w - \frac{\pi}{10}\right) + \delta\left(w + \frac{\pi}{10}\right)\right), -\pi < w < \pi, period = 2\pi$$

Let $w'[k] = w[n-9] = \begin{cases} 1, & 0 \le n \le 18 \\ 0, & \text{else} \end{cases}$

DTFT of $w'[k]$ is $\text{psinc}_{19}(w)e^{-jw(19-1)/2}$, or $\text{psinc}_{19}(w)e^{-jw9}$

Then, by the shift property,

$$w[k] < \frac{DTFT}{\quad} > \text{psinc}_{19}(w)$$

Noting that $w[n-k] = w[k-n]$,

$$w[k-n] < \frac{DTFT}{\quad} > \text{psinc}_{19}(w)e^{-jwn}$$

Hence,

$$\begin{aligned} \tilde{S}(w,n) &= DTFT\left\{\cos\left(\frac{\pi k}{10}\right) w[n-k]\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left(\delta\left(w - \mu - \frac{\pi}{10}\right) + \delta\left(w - \mu + \frac{\pi}{10}\right)\right) psinc_{19}(\mu)e^{-j\mu n}d\mu \\ &= \frac{1}{2}\left(psinc_{19}\left(w - \frac{\pi}{10}\right)e^{-j\left(w-\frac{\pi}{10}\right)n} + psinc_{19}\left(w + \frac{\pi}{10}\right)e^{-j\left(w+\frac{\pi}{10}\right)n}\right) \end{aligned}$$

When $n = -20$,

$$\tilde{S}(w,-20) = \frac{1}{2}\left(psinc_{19}\left(w - \frac{\pi}{10}\right)e^{-j\left(w-\frac{\pi}{10}\right)(-20)} + psinc_{19}\left(w + \frac{\pi}{10}\right)e^{-j\left(w+\frac{\pi}{10}\right)(-20)}\right)$$

(ii) If $n > 10$,

$$x[n] = cos\left(\frac{\pi n}{2}\right)$$

Similarly to (i),

$$\begin{aligned}
\tilde{S}(w,n) &= \sum_k cos\left(\frac{\pi n}{2}\right) w[n-k]e^{-jwk} \\
&= DTFT\left\{\cos\left(\frac{\pi n}{2}\right) w[n-k]\right\} \\
&= \frac{1}{2}\left(psinc_{19}\left(w-\frac{\pi}{2}\right)e^{-j\left(w-\frac{\pi}{2}\right)n} + psinc_{19}\left(w+\frac{\pi}{2}\right)e^{-j\left(w+\frac{\pi}{2}\right)n}\right)
\end{aligned}$$

When $n = 20$,

$$\tilde{S}(w,20) = \frac{1}{2}\left(psinc_{19}\left(w-\frac{\pi}{2}\right)e^{-j\left(w-\frac{\pi}{2}\right)(20)} + psinc_{19}\left(w+\frac{\pi}{2}\right)e^{-j\left(w+\frac{\pi}{2}\right)(20)}\right)$$

(iii) If $n = 0$,

$$\begin{aligned}
\tilde{S}(w,0) &= \sum_k x[k]w[-k]e^{-jwk} \\
&= \sum_k x[k]w[k]e^{-jwk} \quad \text{(by symmetry)} \\
&= \sum_{k=-9}^{-1} cos\left(\frac{\pi n}{10}\right)e^{-jwk} + \sum_{k=0}^{9} cos\left(\frac{\pi n}{2}\right)e^{-jwk} \\
&= \left(\sum_{k=-9}^{0} cos\left(\frac{\pi n}{10}\right)e^{-jwk} - 1\right) + \sum_{k=0}^{9} cos\left(\frac{\pi n}{2}\right)e^{-jwk} \\
&= \sum_k cos\left(\frac{\pi n}{10}\right)w_1[-k]e^{-jwk} + \sum_k cos\left(\frac{\pi n}{2}\right)w_1[k]e^{-jwk} - 1
\end{aligned}$$

where

$$w_1[k] = \begin{cases} 1, 0 \le k \le 9 \\ 0, \quad \text{else} \end{cases}$$

$$w_1[k] < \xrightarrow{DTFT} > \mathrm{psinc}_{10}(w)e^{-jw\frac{9}{2}}$$

$$\begin{aligned}
w_1[-k] < \xrightarrow{DTFT} > &\ \mathrm{psinc}_{10}(-w)e^{jw\frac{9}{2}} \\
&= \mathrm{psinc}_{10}(w)e^{jw\frac{9}{2}}
\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{S}(w,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left(\delta\left(w-\mu-\frac{\pi}{10}\right)+\delta\left(w-\mu+\frac{\pi}{10}\right)\right) psinc_{10}(\mu)e^{-j\mu\frac{9}{2}}d\mu \\ &\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi}\pi\left(\delta\left(w-\mu-\frac{\pi}{2}\right)+\delta\left(w-\mu+\frac{\pi}{2}\right)\right)psinc_{10}(\mu)e^{-j\mu\frac{9}{2}}d\mu - 1 \\ &= \frac{1}{2}\left(psinc_{10}\left(w-\frac{\pi}{10}\right)e^{-j\left(w-\frac{\pi}{10}\right)\left(-\frac{9}{2}\right)} + psinc_{10}\left(w+\frac{\pi}{10}\right)e^{-j\left(w+\frac{\pi}{10}\right)\left(-\frac{9}{2}\right)}\right) \\ &\quad +\frac{1}{2}\left(psinc_{10}\left(w-\frac{\pi}{2}\right)e^{-j\left(w-\frac{\pi}{2}\right)\left(\frac{9}{2}\right)}+psinc_{10}\left(w+\frac{\pi}{2}\right)e^{-j\left(w+\frac{\pi}{2}\right)\left(\frac{9}{2}\right)}\right)-1\end{aligned}$$

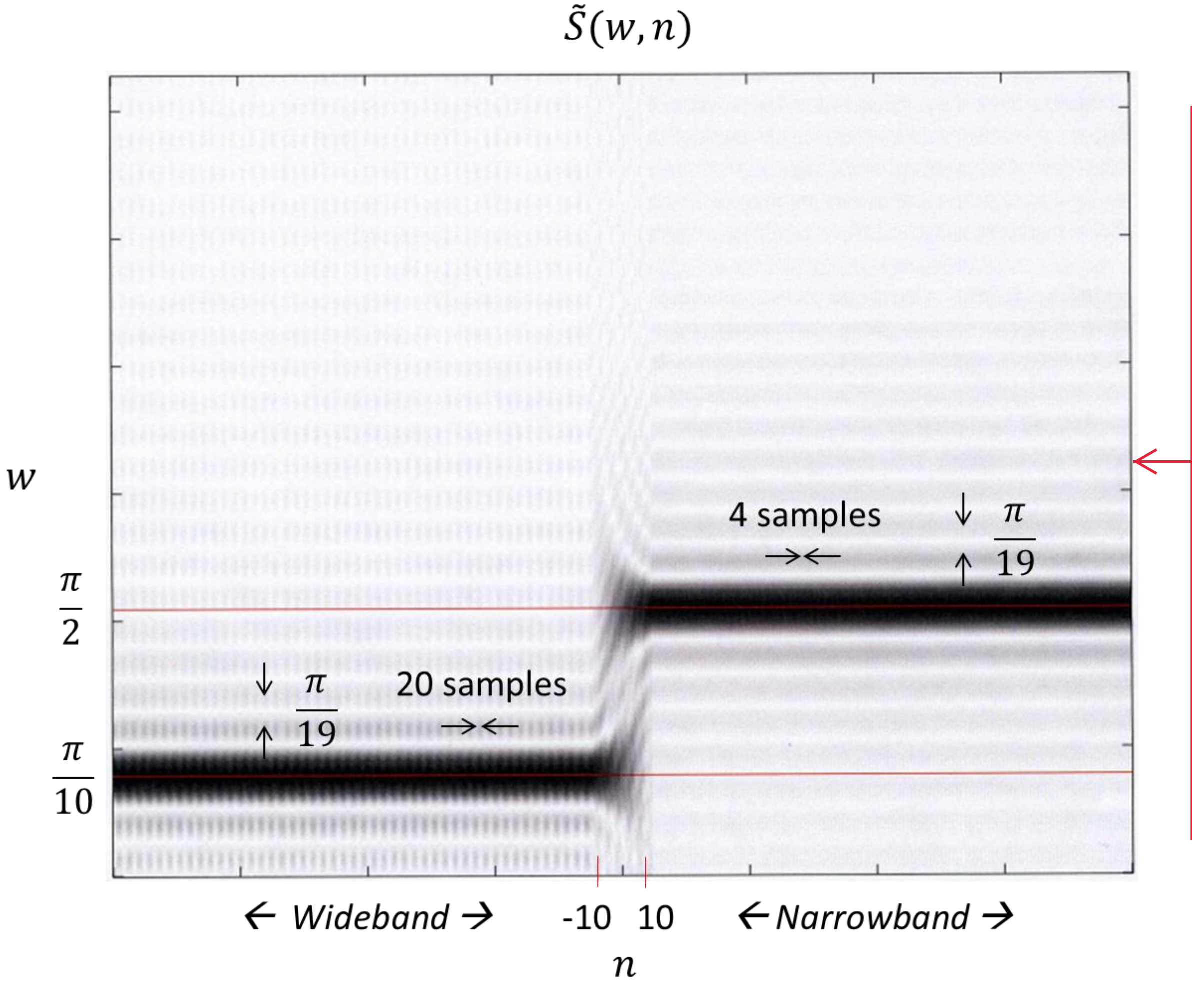
b. (7) Sketch $\left|\tilde{S}(w,n)\right|$ for $0 \leq w \leq \pi$ and for all n. Be sure to label and dimension all important quantities.

Notice that the window length is 19 samples and you have several shifted $psinc_{19}$, which have peaks at $w = 0, \pm\frac{\pi}{19}, \pm\frac{3\pi}{19}, \ldots$

For $w = \frac{\pi}{2}$, the period of $\cos\left(\frac{\pi n}{2}\right)$ is 4 samples. The spectrogram does not show good resolution of the period of the waveform, and is effectively narrowband for $n > 10$.

In contrast, for $w = \frac{\pi}{10}$, the period of $\cos\left(\frac{\pi n}{10}\right)$ is 20 samples. We can see vertical striations suggesting that this is a wideband spectrogram for $n < -10$.

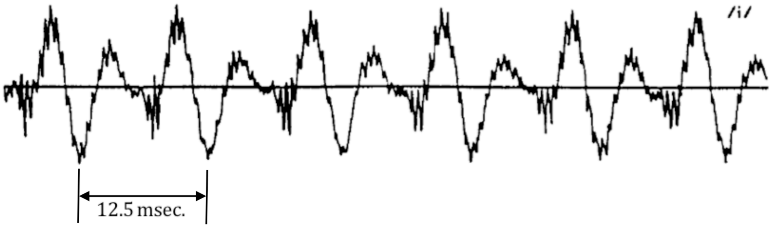
Note the spreading of the spectral peaks at the window becomes narrower on either side of the origin.



Note that these are sidelobes of psinc_19(.).

They are not spaced apart by the pitch period, since this waveform is a single cosine, and is not the vocal tract response repeated with the pitch period.

4. (25) Consider a portion of the waveform for the voiced phoneme /i/ shown below:



a. (5) From the table below, identify the first and second formant frequencies for this phoneme.

Table 3.2 Average Formant Frequencies for the Vowels. (After Peterson and Barney [11].)

FORMANT FREQUENCIES FOR THE VOWELS					
Typewritten Symbol for Vowel	IPA Symbol	Typical Word	F_1	F_2	F_3
IY	i	(beet)	270	2290	3010
I	ɪ	(bit)	390	1990	2550
E	ɛ	(bet)	530	1840	2480
AE	æ	(bat)	660	1720	2410
UH	ʌ	(but)	520	1190	2390
A	ɑ	(hot)	730	1090	2440
OW	ɔ	(bought)	570	840	2410
U	ʊ	(foot)	440	1020	2240
OO	u	(boot)	300	870	2240
ER	ɝ	(bird)	490	1350	1690

Consider the STDTFT of the speech waveform defined as

$$\tilde{S}(\omega,n)=\sum_{k=-\infty}^{\infty} s[k]w[n-k]e^{-jwk}\ .$$

Assume that the speech waveform shown above is sampled at an 8 kHz rate.

b. (5) Choose an appropriate length for the window function $w[n]$ that will yield a *narrowband* spectrogram.

c. (5) Sketch what this narrowband spectrogram would look like. Be sure to label and dimension all important quantities.

d. (5) Choose an appropriate length for the window function $w[n]$ that will yield a *wideband* spectrogram.

e. (5) Sketch what this wideband spectrogram would look like. Be sure to label and dimension all important quantities.

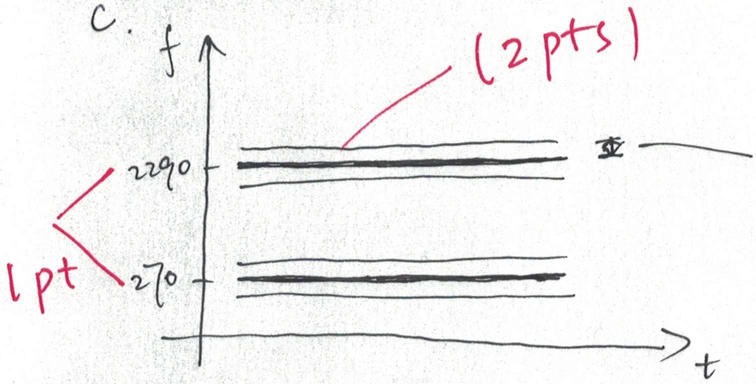
4. (continued - 1)

a. $F_1 = 270$ Hz , $F_2 = 2290$ Hz (5 pt)

b. $P = 12.5$ ms × 8 kHz = 100 samples

Choose $N >> P \Rightarrow N = 500$ (5 pt)

c.

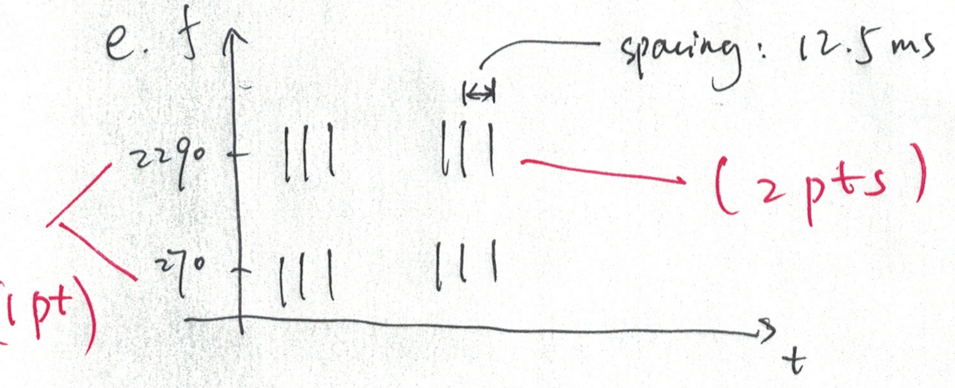


spacing : $\frac{1}{12.5ms}$ = 80 Hz (1 pt)

Axis Label : (1 pt)

d. Choose $N < P \Rightarrow N = 50$ (5 pt)

e.



Axis Label : (1 pt)