21 March 2018 **Name:___________________**

ECE 438 **Exam No. 2** **Spring 2018**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the Z-transform $X(z)$, given below, of a DT signal $x[n]$.

$$X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + 2z^{-1}\right)}$$

a. (6) Find the three possible regions of convergence for $X(z)$.

b. (19) For each possible region of convergence for $X(z)$, find the corresponding signal $x[n]$.

2. (25) Fast Fourier Transform Algorithm

   a. (13) Derive the complete equations that describe a Fast Fourier Transform (FFT) Algorithm to compute a 6-point Discrete Fourier Transform (DFT).

   b. (12) Draw a complete and fully labeled flow diagram for your 6-point FFT algorithm.

3. (25 pts).

a) (5) Let $x[n]=e^{j2\pi(2)n/5}, n=0,1,2,3,4$. Find the 5-point DFT $X[k], k=0,1,2,3,4$ of $x[n]$.

b) (2) Carefully sketch $X[k], k=0,1,2,3,4$ from your answer to part (a) above.

c) (12) Let $x[n]=e^{j2\pi(5)n/10}, n=0,1,2,3,4$. Find the 5-point DFT $X[k], k=0,1,2,3,4$ of $x[n]$.

d) (6) Carefully sketch $X[k], k=0,1,2,3,4$ from your answer to part (c) above.

4. (25 pts) Consider the two signals $x_1[n]$ and $x_2[n]$ given below.

n	0	1	2
$x_1[n]$	1	1	1

n	0	1	2
$x_2[n]$	$\frac{1}{2}$	1	$\frac{1}{2}$

a) (11) Find the aperiodic convolution of $x_1[n]$ and $x_2[n]$.

b) (12). Find the 3-point periodic convolution of $x_1[n]$ and $x_2[n]$.

c) (2) To what length N would the signals $x_1[n]$ and $x_2[n]$ need to be zero-padded so that a portion of the N-point periodic convolution of $x_1[n]$ and $x_2[n]$ matches the non-zero portion of their aperiodic convolution?

Question 1

(25 pts.) Consider the Z-transform $X(z)$, given below, of a DT signal $x[n]$.

$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})}$$

1. (6) Find the three possible regions of convergence for $X(z)$.

   $$X(z) = \frac{z^2}{(z-\frac{1}{2})(z+2)}$$

   Therefore, there are three possible regions:
   1) (**2 pts.**) $|z| < \frac{1}{2}$ and $|z| < 2 ==> |z| < \frac{1}{2}$
   2) (**2 pts.**) $|z| > \frac{1}{2}$ and $|z| < 2 ==> \frac{1}{2} < |z| < 2$
   3) (**2 pts.**) $|z| > \frac{1}{2}$ and $|z| > 2 ==> |z| > 2$

2. (19) For each possible region of convergence for $X(z)$, find the corresponding signal $x[n]$.

   $$\begin{aligned} X(z) &= \frac{1}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})} \\ &= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+2z^{-1}} \\ &= \frac{A + 2Az^{-1} + B - \frac{1}{2}Bz^{-1}}{(1-\frac{1}{2}z^{-1})(1+2z^{-1})} \end{aligned}$$

   $$\therefore \quad \begin{cases} A + B = 1 \\ 2A - \frac{1}{2}B = 0 \end{cases}$$

   $$or \quad \begin{cases} A = \frac{1}{5} \\ B = \frac{4}{5} \end{cases}$$

   $$\therefore X(z) = \frac{\frac{1}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{5}}{1+2z^{-1}}$$ **(5 pts. for procedure, 2 pts. for results)**

(**4 pts.**) Case (1): $|z| < \frac{1}{2}$

$$x[n] = \frac{1}{5}(-(\frac{1}{2})^n u[-n-1]) + \frac{4}{5}(-(-2)^n u[-n-1])$$

(**4 pts.**) Case (2): $\frac{1}{2} < |z| < 2$

$$x[n] = \frac{1}{5}(\frac{1}{2})^n u[n] + \frac{4}{5}(-(-2)^n u[-n-1])$$

(**4 pts.**) Case (3): $|z| > 2$

$$x[n] = \frac{1}{5}(\frac{1}{2})^n u[n] + \frac{4}{5}(-2)^n u[n]$$

2) $N=6.$

3x2 pt. (2 points)

$$X^6(k) = \sum_{n=0}^{5} x[n] \cdot e^{-j\frac{2\pi kn}{6}}, \quad k = 0, 1, \ldots, 5.$$ (3 points)

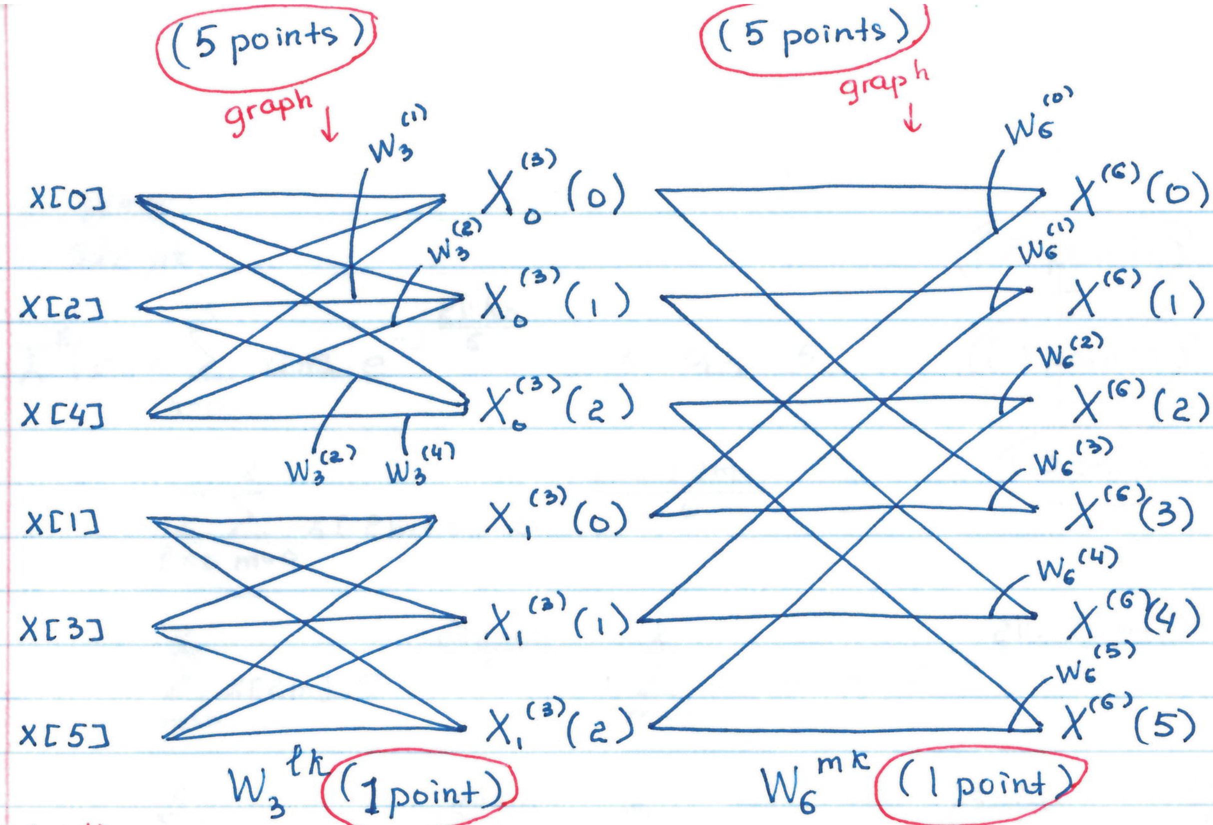
$$= \sum_{\ell=0}^{1} \sum_{m=0}^{2} x[2m+\ell] \cdot e^{-j\frac{2\pi k(2m+\ell)}{6}} =$$ (3 points)

$$= \sum_{m=0}^{2} x[2m] \cdot e^{-j\frac{2\pi k \cdot 2m}{6}} + \sum_{m=0}^{2} x[2m+1] \cdot e^{-j\frac{2\pi k \cdot (2m+1)}{6}} =$$

$$= \sum_{m=0}^{2} x[2m] \cdot e^{-j\frac{2\pi km}{3}} + \sum_{m=0}^{2} x[2m+1] \cdot e^{-j\frac{2\pi km}{3}} \cdot e^{-j\frac{2\pi k}{6}}$$

$$= X_0^{(3)}(k') + X_1^{(3)}(k') \cdot W_6^{(k)}, \text{ where } k' = k \bmod 3 \text{ and } W_N^k = e^{-j\frac{2\pi k}{N}}.$$

(5 points)



Addnl. derivation:

- $X^6(0) = X_0^3(0) + X_1^3(0)\cdot W_6^{(0)}$, $W_6^{(0)} = e^{-j\frac{2\pi 0}{6}} = 1$
- $X^6(1) = X_0^3(1) + X_1^3(1)\cdot W_6^{(1)}$, $W_6^{1} = e^{-j\frac{\pi}{3}}$
- $X^6(2) = X_0^3(2) + X_1^3(2)\cdot W_6^{(2)}$, $W_6^{(2)} = e^{-j\frac{2\pi 2}{6}} = e^{-j\frac{2\pi}{3}}$
- $X^6(3) = X_0^3(0) + X_1^3(0)\cdot W_6^{(3)}$, $W_6^{(3)} = e^{-j\frac{2\pi 3}{6}} = e^{-j\pi}$
- $X^6(4) = X_0^3(1) + X_1^{(3)}(1)\cdot W_6^{(4)}$, $W_6^{(4)} = e^{-j\frac{2\pi 4}{3}} = e^{-j\frac{4\pi}{3}}$.
- $X^6(5) = X_0^3(2) + X_1^{(3)}(2)\cdot W_6^{(5)}$, $W_6^{(5)} = e^{-j\frac{2\pi 5}{6}} = e^{-j\frac{5\pi}{3}}$.

- $X_0^{(3)}(k \bmod 3) = \sum_{m=0}^{2} x[2m]\cdot e^{-j\frac{2\pi k m}{3}}$.

k=0 • $X_0^3(0) = x[0] + x[2]\cdot e^{-j\frac{2\pi 0}{3}} + x[4] = x[0]+x[2]+x[4]$.

k=1 • $X_0^3(1) = x[0]\cdot 1 + x[2]\cdot e^{-j\frac{2\pi 1}{3}} + x[4]\cdot e^{-j\frac{2\pi\cdot 2}{3}} =$

$= x[0] + x[2]\cdot W_3^{(1)} + x[4]\cdot W_3^{(2)}$

k=2 • $X_0^3(2) = x[0] + x[2]\cdot e^{-j\frac{2\pi 2\cdot 1}{3}} + x[4]\cdot e^{-j\frac{2\pi 2\cdot 2}{3}} =$

$= x[0] + x[2]\cdot W_3^{(2)} + x[4]\cdot W_3^{(4)}$

- $X_1^3(0) = \sum_{m=0}^{2} x[2m+1]\cdot e^{-j\cdot 0} = x[1]+x[3]+x[5]$.
- $X_1^{(3)}(1) = \sum_{m=0}^{2} x[2m+1]\cdot e^{-j\frac{2\pi m}{3}} = x[1]\cdot W_3^{(0)} + x[3]\cdot W_3^{(1)} + x[5]\cdot W_3^{(2)}$.
- $X_1^{(3)}(2) = \sum_{m=0}^{2} x[2m+1]\cdot e^{-j\frac{2\pi m\cdot 2}{3}} = x[1]\cdot W_3^{(0)} + x[3]\cdot W_3^{(2)} + x[5]\cdot W_3^{(4)}$.

ECE438 Exam 2 Question 3 Solution

3. a) 6pts

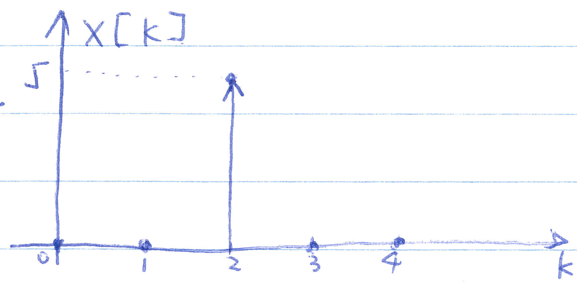
Since $x[n] = e^{j2\pi(2)n/5}$

and we know that

$$e^{j2\pi k_0 n/N},\ 0 \le n \le N-1 \overset{DFT}{\longleftrightarrow} N\delta[k-k_0] \quad \text{where } 0 \le k \le N-1$$ 4 points

So, $X[k] = 5\,\delta[k-2]$, $0 \le k \le 4$ 2 points

b) 3pts



c) 10pts

$x[n] = e^{j2\pi(5)n/10}$, $n = 0, 1, 2, 3, 4$

$w_0 = \frac{2\pi(5)}{10} = \pi$, $N = 5$ 2 points

2 points $X(w) = rep_{2\pi}\{\pi\delta(w-w_0)\}$, $W(w) = e^{-jw\frac{N-1}{2}} \cdot \sin(wN/2)/\sin(w/2)$, $(N=5)$ 2 points

DTFT of $e^{jw_0 n}$, $n = 0, 1, \ldots, N-1$ can be calculated as:

$$X(w) = 2\pi\, rep_{2\pi}\{\delta(w-w_0)\} * e^{-jw\frac{N-1}{2}} \cdot \frac{\sin(wN/2)}{\sin(w/2)}$$ 2 points

Then, we have DFT of $e^{jw_0 n}$:

$$X[k] = \frac{\sin[(\frac{2\pi k}{N} - \pi)\frac{N}{2}]}{\sin[(\frac{2\pi k}{N} - \pi)\cdot\frac{1}{2}]} \cdot e^{-j(\frac{2\pi k}{N} - \pi)\cdot\frac{N-1}{2}}$$

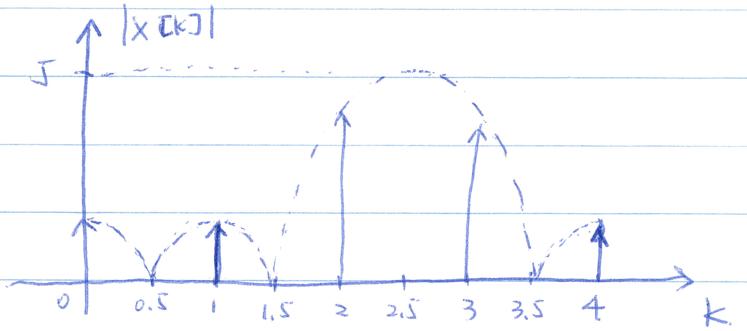
with $N=5$, we have:

$$X[k] = \frac{\sin[(\frac{2\pi k}{5} - \pi)\cdot\frac{5}{2}]}{\sin[(\frac{2\pi k}{5} - \pi)\cdot\frac{1}{2}]} \cdot e^{-j(\frac{2\pi k}{5} - \pi)\cdot 2}$$ 2 points

$$= psinc_5\left(\frac{2\pi k}{5} - \pi\right) \cdot e^{-j(\frac{2\pi k}{5} - \pi)\cdot 2}$$

$k = 0, 1, 2, 3, 4$

d)
6pts



Question 4

n	0	1	2
$x_1[n]$	1	1	1

n	0	1	2
$x_2[n]$	1/2	1	1/2

a) Find the aperiodic convolution

$$y_1[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=0}^{2} x_1[k]\, x_2[n-k] = x_2[n] + x_2[n-1] + x_2[n-2]$$

3 points

n	0	1	2	3	4	5
$x_2[n]$	1/2	1	1/2			
$+x_2[n-1]$		1/2	1	1/2		
$+x_2[n-2]$			1/2	1	1/2	
$y_1[n]$	1/2	3/2	2	3/2	1/2	0

6 points for correct result
2 points for correct n

no points deducted if there is no equation, with correct result

b) Find the 3-point periodic convolution of $x_1[n]$ and $x_2[n]$

n	0	1	2
$x_2[n \bmod 3]$	1/2	1	1/2
$x_2[n-1 \bmod 3]$	1/2	1/2	1
$x_2[n-2 \bmod 3]$	1	1/2	1/2
$y_2[n]$	2	2	2

$$y_2[n] = x_1[n] \circledast x_2[n] = \sum_{k=0}^{2} x[k] \cdot x_2[(n-k) \bmod 3]$$

3 points

$$= x_2[n \bmod 3] + x_2[(n-1) \bmod 3] + x_2[(n-2) \bmod 3]$$

if the result was wrong, then gave 2-4 points for the table

c) To what length N would the signals $x_1[n]$ and $x_2[n]$ need to be zero-padded so that a portion of the N-point periodic convolution of $x_1[n]$ and $x_2[n]$ matches the non-zero portion of their aperiodic convolution?

If $x_1[n]$ has length M_1 and $x_2[n]$ has length M_2, periodic convolution with length $M_1 + M_2 - 1$ will match the aperiodic convolution results.

$M_1 = 3$, $M_2 = 3$ => $M_1 + M_2 - 1 = 5$ 2 points

Therefore, signals $x_1[n]$ and $x_2[n]$ need to be zero-padded with two zeros to length $N = 5$