14 February 2018 **Name:____________________**

ECE 438 **Exam No. 1** **Spring 2018**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

a. (9) Find the response of this system to the input

$$x[n] = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{else} \end{cases}.$$

Assume that the system is initially at rest. So $y[n] = 0, n < 0$.

b. (9) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (7) Based on your answer to part (b), find simple expressions for the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

2. (25 pts.) Perform the convolution $w[n]$ of the following two signals, and carefully sketch the output signal $w[n]$.

$$x[n] = \left(\frac{1}{2}\right)^n u[n], \text{ and } y[n] = \begin{cases} 1, & 0 \le n \le 9 \\ 0, & \text{else} \end{cases} .$$

3. (25) Consider the continuous-time signal $x(t) = \cos\left(2\pi(3000)t\right) + \cos\left(2\pi(7000)t\right)$.

a. (8) Find a simple expression for the CTFT $X(f)$ of $x(t)$, and sketch it.

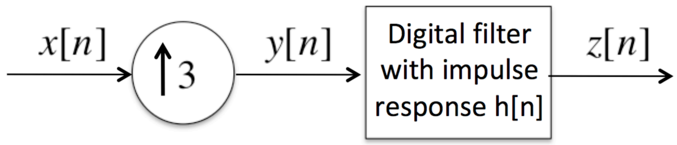
Suppose that we sample $x(t)$ at an 8 kHz sampling rate to generate the continuous-time sampled signal $x_s(t) = \text{comb}_T\left[x(t)\right]$, where $T = 1/8000$ sec. is the sampling interval.

b. (9) Find a simple expression for the CTFT $X_s(f)$ of $x_s(t)$, and sketch it. Your final answer should not contain any operators.

Now, suppose that we input $x_s(t)$ to an ideal low-pass reconstruction filter with frequency response $H_r(f) = T \cdot \text{rect}(Tf)$.

c. (8) Find a simple expression for the output $x_r(t)$ from the reconstruction filter. Your final answer should not contain any operators.

4. (25 pts) Consider the system shown below:



The impulse response of the filter is given by:

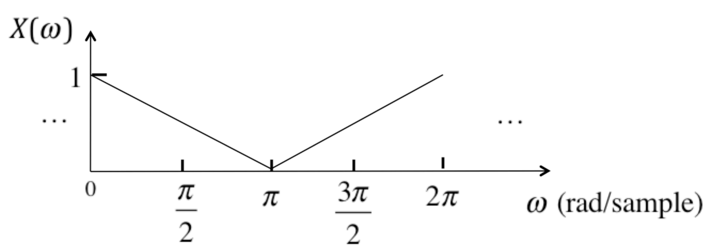
$$h[n] = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{else} \end{cases} .$$

a) (6) Find the response $z[n]$ of this system to the input $x[n]$ given below:

n	…	-1	0	1	2	3	…
$x[n]$	…	0	1	2/3	1/3	0	…

b) (13) Find a simple closed-form expression for the DTFT $Z(\omega)$ of the output $z[n]$ in terms of the DTFT $X(\omega)$ of the input $x[n]$. (Note that this expression should be valid for any input $x[n]$.) Your answer should not contain any operators.

c) (6) Sketch $Z(\omega)$, when $X(\omega)$ is given as below:



Be sure to dimension all important quantities.

ECE438 Exam I Q1 Solution

1. (a)

n	$x[n]$	$x[n-1]$	$y[n-1]$	$y[n]$
-1	0	0	0	0
0	1	0	0	1
1	1	1	1	-1
2	1	1	-1	1
3	0	1	1	-2
4	0	0	-2	2
5	0	0	2	-2
				⋮

6 pts

$$y[n] = \begin{cases} (-1)^n & n = 0, 1, 2 \\ (-1)^n \cdot 2 & n \geq 3 \end{cases}$$

3 pts

(b)

$$Y(\omega) = X(\omega) - X(\omega)e^{-j\omega} - Y(\omega)e^{-j\omega}$$ 3 pts

$$Y(\omega)(1 + e^{-j\omega}) = X(\omega)(1 - e^{-j\omega})$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}$$ 2 pts

$$= \frac{e^{-\frac{j\omega}{2}}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})}$$ 2 pts

$$= \frac{j\sin(\omega/2)}{\cos(\omega/2)} = j\tan(\omega/2)$$ 2 pts

(c)

$$|H(\omega)| = |j\tan(\tfrac{\omega}{2})| = |\tan(\tfrac{\omega}{2})|$$ 2 pts

$$\angle H(\omega) = \angle j + \angle \tan\frac{\omega}{2}$$ 3 pts

$$= \begin{cases} \frac{\pi}{2} + 0 & 0 < \omega < \pi \\ \frac{\pi}{2} - \pi & -\pi < \omega < 0 \end{cases}$$

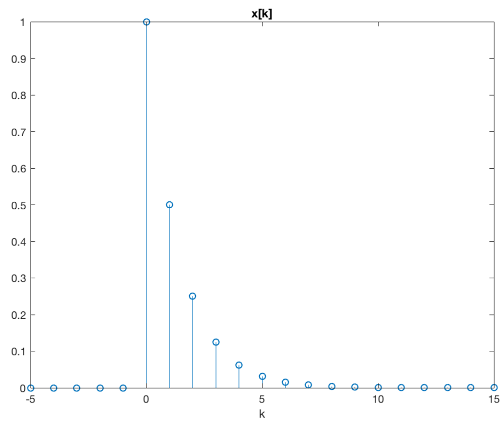
$$= \begin{cases} \frac{\pi}{2} & 0 < \omega < \pi \\ -\frac{\pi}{2} & -\pi < \omega < 0 \end{cases}$$ 2 pts

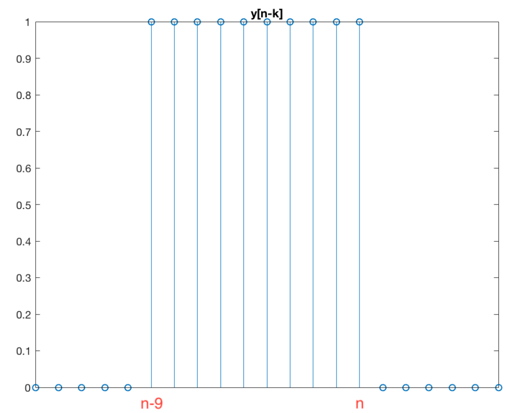
Question 2

$$x[n] = (\frac{1}{2})^n u[n]$$

$$y[n] = \begin{cases} 1 & 0 \leqslant n \leqslant 9 \\ 0 & else \end{cases}$$

$$\text{let } w[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$





$\underline{n < 0:}$ 2 pts - condition

$w[n] = 0$ 3 pts - result

$\underline{n - 9 < 0, n \geqslant 0 => 0 \leqslant n < 9:}$ 3 pts - condition

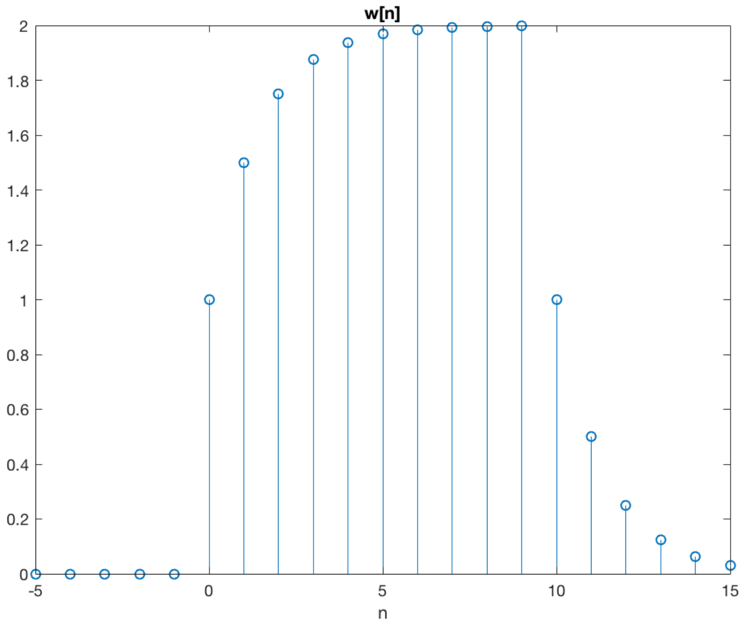
$$w[n] = \sum_{k=0}^{n}(\frac{1}{2})^k = \frac{1-(\frac{1}{2})^{n+1}}{\frac{1}{2}} = 2 - (\frac{1}{2})^n$$

2 pts - correct form (sum of (1/2)^k)
2 pts - boundary
1 pt - result

$\underline{n \geqslant 9:}$ 3 pts - condition

$$w[n] = \sum_{k=n-9}^{n}(\frac{1}{2})^k = \sum_{k=0}^{n}(\frac{1}{2})^k - \sum_{k=0}^{n-10}(\frac{1}{2})^k = \frac{1-(\frac{1}{2})^{n+1}}{\frac{1}{2}} - \frac{1-(\frac{1}{2})^{n-9}}{\frac{1}{2}} = (\frac{1}{2})^{n-10} - (\frac{1}{2})^n$$

2 pts - correct form (sum of (1/2)^k)
2 pts - boundary
3 pts - result



1 pt - graph based on student's answer

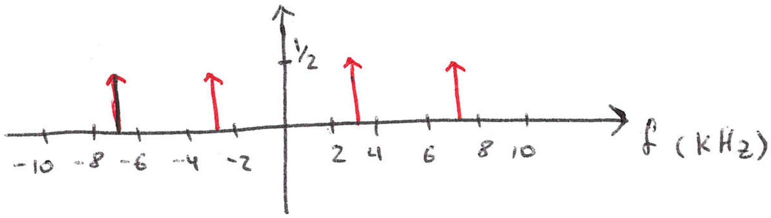
1 pt - correct result

Question 3

$x(t) = \cos(2\pi(3000)t) + \cos(2\pi(7000)t)$

a) Using CTFT pairs:

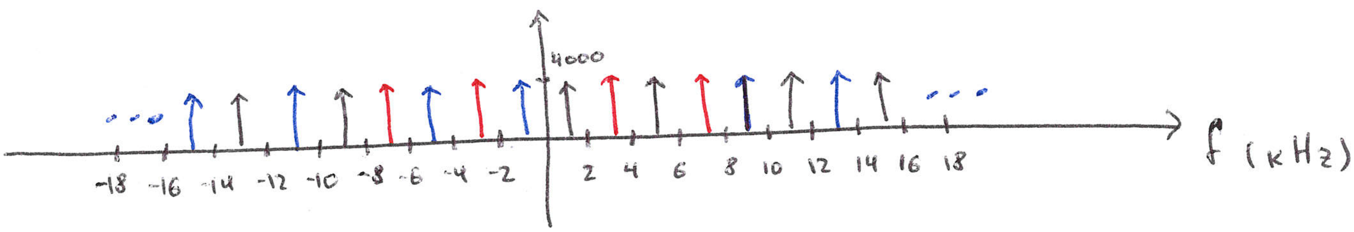
$X(f) = \frac{1}{2}\{\delta(f-3000) + \delta(f+3000) + \delta(f-7000) + \delta(f+7000)\}$



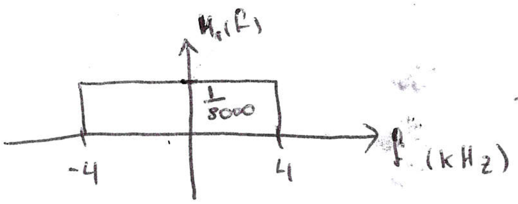
b) $x_s(t) = comb_T[x(t)]$ $T = 1/8000$ sec

$X_s(f) = \frac{1}{T} rep_{1/T}[X(f)] = 8000\, rep_{8000}[X(f)] = \sum_{k=-\infty}^{\infty} 4000\{\delta(f-3000-8000k) +$

$+\delta(f+3000-8000k) + \delta(f-7000-8000k) + \delta(f+7000-8000k)\}$



c) $H_r(f) = T \cdot rect(Tf) = \frac{1}{8000} rect(\frac{f}{8000})$



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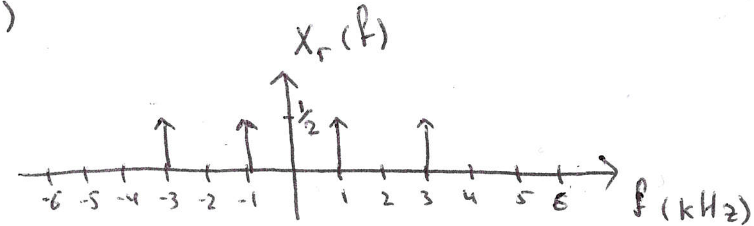


$X_r(f) = \frac{1}{2}\{\delta(f+3000) + \delta(f+1000) + \delta(f-1000) + \delta(f-3000)\}$

$x_r(t) = \cos(2\pi(3000)t) + \cos(2\pi(1000)t)$

#4. (a)

$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$x[n] = \delta[n] + \frac{2}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$

(Step 1) After upsampling,

$y[n] = \delta[n] + \frac{2}{3}\delta[n-3] + \frac{1}{3}\delta[n-6]$ … 3 pts (Step 1)

(Step 2 and 3) After filtering with $h[n]$,

$z[n] = y[n] * h[n]$ … 1 pt (Step 2 – method 1)

$= \delta[n] + \delta[n-1] + \delta[n-2] + \frac{2}{3}(\delta[n-3] + \delta[n-4] + \delta[n-5])$
$+ \frac{1}{3}(\delta[n-6] + \delta[n-7] + \delta[n-8])$ … 2 pts (Step 3 – method 1)

or

DTFT of each signal or filter gives

$H(w) = 1 + e^{-jw} + e^{-j2w}$

$Y(w) = 1 + \frac{2}{3}e^{-j3w} + \frac{1}{3}e^{-j6w}$

$Z(w) = Y(w)H(w)$ … 1 pt (Step 2 - method 2)

$= 1 + e^{-jw} + e^{-j2w} + \frac{2}{3}\left(e^{-j3w} + e^{-j4w} + e^{-j5w}\right) + \frac{1}{3}\left(e^{-j6w} + e^{-j7w} + e^{-j8w}\right)$

Inverse DTFT of $Z(w)$ gives

$z[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \frac{2}{3}(\delta[n-3] + \delta[n-4] + \delta[n-5])$
$+ \frac{1}{3}(\delta[n-6] + \delta[n-7] + \delta[n-8])$ … 2 pts (Step 3 – method 2)

#4. (b)

(Step 1)

$H(w) = 1 + e^{-jw} + e^{-j2w}$ ... 4 pts (Step 1 – method 1)

or

$$H(w) = \frac{1 - e^{-j3w}}{1 - e^{-jw}} = \frac{e^{\frac{j3w}{2}} - e^{-\frac{j3w}{2}}}{e^{\frac{jw}{2}} - e^{-\frac{jw}{2}}} \cdot \frac{e^{-\frac{j3w}{2}}}{e^{-\frac{jw}{2}}} = \frac{\sin\left(\frac{3w}{2}\right)}{\sin\left(\frac{w}{2}\right)} e^{-jw}$$

$= psinc_3(w)e^{-jw}$ ... 4 pts (Step 1 – method 2)

or

$H(w) = \left(\mathrm{e}^{\mathrm{jw}} + 1 + e^{-jw}\right)e^{-jw}$

$= (1 + 2\cos(w))e^{-jw}$ ... 4 pts (Step 1 – method 3)

(Step 2)

$Z(w) = Y(w)H(w)$

$= X(3w)H(w)$ ... 4 pts (Step 2)

(Step 3)

Therefore,

$Z(w) = X(3w)\left(1 + e^{-jw} + e^{-j2w}\right)$ ... 5 pts (Step 3 – method 1)

or

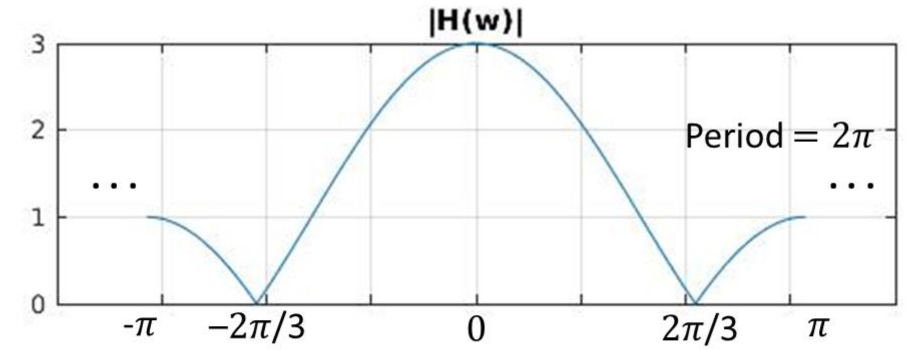
$Z(w) = X(3w)psinc_3(w)e^{-jw}$ ... 5 pts (Step 3 – method 2)

or

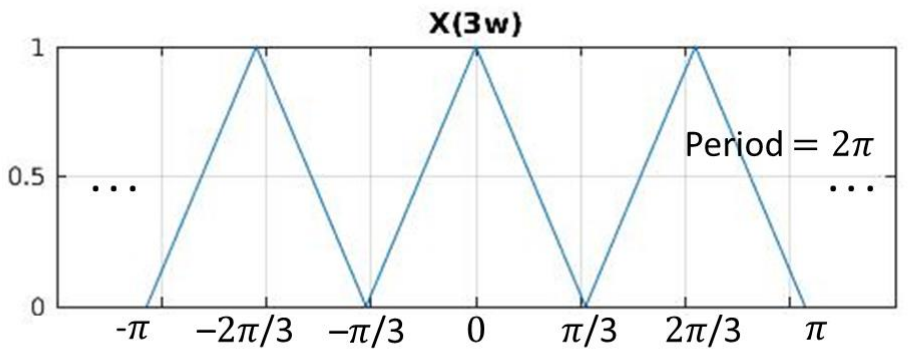
$Z(w) = X(3w)(1 + 2\cos(w))e^{-jw}$ ... 5 pts (Step 3 – method 3)

#4. (c)

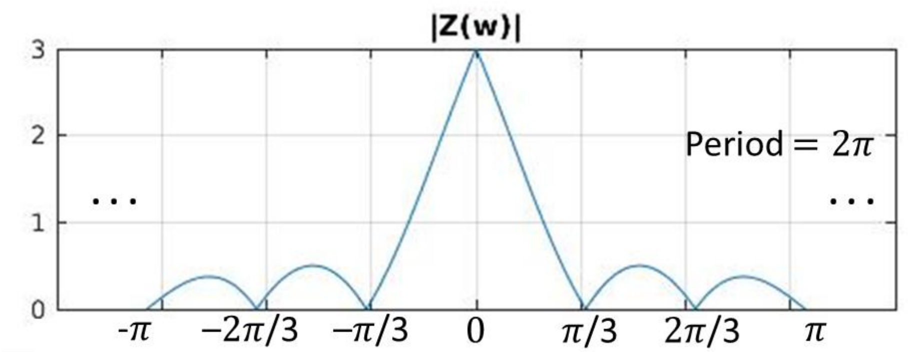
$$|Z(w)| = |Y(w)H(w)| = |X(3w)H(w)| = |X(3w)||H(w)|$$



... 2 pts (Step 1)



... 2 pts (Step 2)



... 2 pts (Step 3)

- It is important to specify that the signal is periodic with a period of 2π. Answers without such do not get full credits.
- 2 points are given if rough shape is the same without a proper procedure.