5 May 2016 **Name:**_______________ **Signature:** _________________

ECE 438 **Final Exam** **Spring 2016**

- You have 120 minutes to work the following five problems.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem. Please note that the problems are arranged in the order that the topics were covered during the semester, not necessarily in the order of difficulty to solve them.

1. (25 pts.) Consider a system described by the following equation

$$y[n]=\frac{1}{2}x[n]+y[n-1].$$

a. (8) Find the response $y[n]$ to the following input

$$x[n]=\begin{cases}1, & n\ge 0\\ 0, & \text{else}\end{cases}.$$

b. (8) Find a simple expression for the frequency response $H(\omega)$ for this system.

c. (6) From your answer to part (b), determine simple expressions for the magnitude and phase of the frequency response.

d. (3) Is this system bounded-input-bounded-output stable?

a.

n	$x[n]$	$y[n]$	$y[n-1]$
-1	0	0	0
0	1	$1/2$	0
1	1	1	$1/2$
2	1	$3/2$	1
3	1	2	$3/2$
⋮	⋮	⋮	⋮

$\therefore y[n]=\frac{n+1}{2}u[n]$

1. (continued – 1)

b. $Y(\omega) = \frac{1}{2}X(\omega) + Y(\omega)e^{-j\omega}$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/2}{1-e^{-j\omega}}$$

$$= \frac{1}{2(1-e^{-j\omega})} = \frac{e^{j\omega/2}}{2(e^{j\omega/2}-e^{-j\omega/2})}$$

$$= \frac{e^{j\frac{\omega}{2}}}{2(2j\sin\frac{\omega}{2})} = \frac{e^{j\omega/2}}{4j\sin\omega/2}$$

c. $$|H(\omega)| = \frac{1}{|4j|}\left|e^{j\frac{\omega}{2}}\right|\left|\frac{1}{\sin\frac{\omega}{2}}\right| = \frac{1}{\left|4\sin\frac{\omega}{2}\right|}$$

$$\angle H(\omega) = \angle\frac{1}{4j} + \angle e^{j\frac{\omega}{2}} + \angle\frac{1}{\sin\frac{\omega}{2}}$$

$$= -\frac{\pi}{2} + \frac{\omega}{2} + \angle\frac{1}{\sin\omega/2}$$

$$= \begin{cases} \frac{\omega}{2} - \frac{\pi}{2} + 0 = \frac{\omega}{2} - \frac{\pi}{2}, & 0 < \omega < 2\pi \\ \frac{\omega}{2} - \frac{\pi}{2} + \pi = \frac{\omega}{2} + \frac{\pi}{2}, & -2\pi < \omega < 0 \end{cases}$$

d. No. $\because$ for a ~~input~~ bounded input, there is not a "B" such that $y[n] \le |B|$ when $n \to \infty$.

2. (25 pts.) A discrete-time LTI system has transfer function $H(z)=\dfrac{z+1}{z^2+1/2}$.

   a. (5) Sketch the locations of the poles and zeros for this system in the complex z-plane.

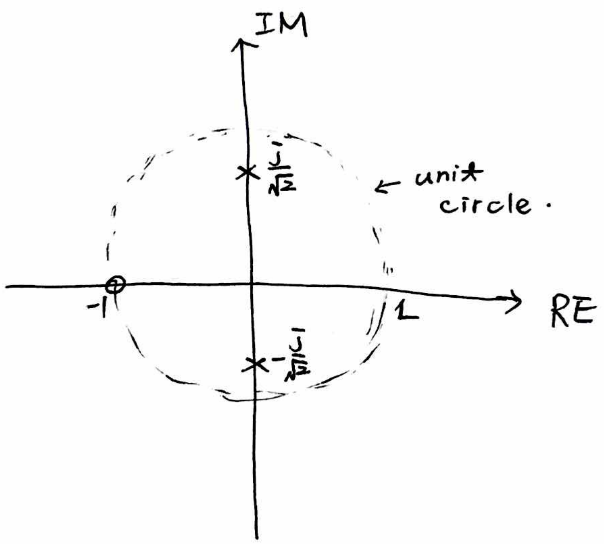
   b. (15) Using the graphical approach, find the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ at the frequencies $\omega = 0, \pi/4, \pi/2$, and π radians/sample. You need only simplify your answers to the extent that is reasonable without the aid of a calculator.

   c. (5) Find a difference equation for $y[n]$ in terms of the current and previous inputs and previous outputs that could be used implement this system.

a) $H(z) = \dfrac{z+1}{z^2+\frac{1}{2}} = \dfrac{z+1}{(z+\frac{1}{\sqrt{2}}j)(z-\frac{1}{\sqrt{2}}j)}$ ← zero ← poles.

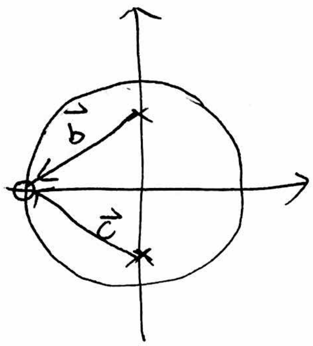
So Poles is at $z = \pm j\frac{1}{\sqrt{2}}$

zeros is at $z = -1$



2. (continued – 2)

$\omega = \pi$:



$\vec{a} = 0$, $|\vec{a}| = 0$, $\angle \vec{a} = 0$

$\vec{b} = -1 - \frac{j}{\sqrt{2}}$, $\angle \vec{b} = -\angle \vec{c}$

$\vec{c} = -1 + \frac{j}{\sqrt{2}}$,

$|H(\omega)| = 0$, $\angle H(\omega) = 0$

c. $$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 + \frac{1}{2} z^{-2}} = \frac{z^{-1}(1 + z^{-1})}{1 + \frac{1}{2} z^{-2}}$$

$$\Rightarrow Y(z)(1 + \tfrac{1}{2} z^{-2}) = X(z) z^{-1} (1 + z^{-1})$$

$$\Rightarrow Y(z) + \tfrac{1}{2} z^{-2} Y(z) = z^{-1} X(z) + z^{-2} X(z)$$

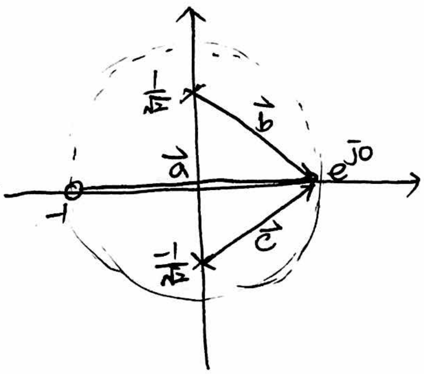
$$\Rightarrow y[n] + \tfrac{1}{2} y[n-2] = x[n-1] + x[n-2]$$

Thus, the differential equation for $y[n]$ is,

$$y[n] = x[n-1] + x[n-2] - \tfrac{1}{2} y[n-2]$$

2. (continued – 1)

(b) $\omega = 0$:



$\vec{a} = 2$, $|\vec{a}| = 2$, $\angle\vec{a} = 0$

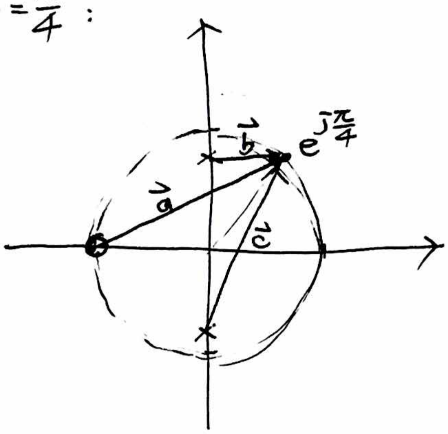
$\vec{b} = 1 - \frac{1}{\sqrt{2}}j$, $|\vec{b}| = \sqrt{1 + (\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{3}{2}}$, $\angle\vec{b} = -\angle\vec{c}$

$\vec{c} = 1 + \frac{1}{\sqrt{2}}j$ $|\vec{c}| = \sqrt{1 + (\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{3}{2}}$

$$|H(e^{j0})| = \frac{|\vec{a}|}{|\vec{b}||\vec{c}|} = \frac{2}{\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}} = \frac{4}{3}$$

$$\angle H(e^{j0}) = \angle\vec{a} - \angle\vec{b} - \angle\vec{c} = 0 - \angle\vec{b} - (-\angle\vec{b}) = 0$$

$\omega = \frac{\pi}{4}$:



$\vec{a} = (1 + \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}}j$, $|\vec{a}| = \sqrt{(1+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2}$,

$\vec{b} = \frac{1}{\sqrt{2}}$ $= \sqrt{2 + \sqrt{2}}$

$\vec{c} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}j$ $|\vec{b}| = \frac{1}{\sqrt{2}}$

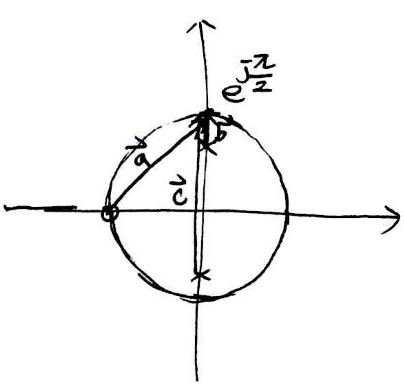
$|\vec{c}| = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{2}{\sqrt{2}})^2} = \sqrt{\frac{5}{2}}$

$$\angle\vec{a} = \tan^{-1}\frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} = \tan^{-1}\left(\frac{1}{\sqrt{2}+1}\right)$$

$$\angle\vec{b} = 0 , \quad \angle\vec{c} = \tan^{-1}\frac{\frac{2}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \tan^{-1}(2)$$

$$|H(e^{j\frac{\pi}{4}})| = \frac{|\vec{a}|}{|\vec{b}||\vec{c}|} = \frac{2\sqrt{2+\sqrt{2}}}{\sqrt{5}} , \quad \angle H(e^{j\frac{\pi}{4}}) = \tan^{-1}\left(\frac{1}{\sqrt{2}+1}\right) - 0 - \tan^{-1}(2)$$

$\omega = \frac{\pi}{2}$:



$\vec{a} = 1 + j$ $|\vec{a}| = \sqrt{2}$, $\angle\vec{a} = \frac{\pi}{4}$

$\vec{b} = (1 - \frac{1}{\sqrt{2}})j$ $|\vec{b}| = 1 - \frac{1}{\sqrt{2}}$, $\angle\vec{b} = \frac{\pi}{2}$

$\vec{c} = (1 + \frac{1}{\sqrt{2}})j$ $|\vec{c}| = 1 + \frac{1}{\sqrt{2}}$, $\angle\vec{c} = \frac{\pi}{2}$

$$|H(\omega)| = \frac{\sqrt{2}}{(1-\frac{1}{\sqrt{2}})(1+\frac{1}{\sqrt{2}})} = 2\sqrt{2}$$

$$\angle H(\omega) = \frac{\pi}{4} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3\pi}{4}$$

3. (25 pts.) Consider a random variable X with density function

$$f_X(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & \text{else} \end{cases}.$$

a. (6) Find the mean and variance of X.

Suppose we generate a new random variable $Y = |X|$.

b. (7) Find the mean and variance of Y.

c. (7) Determine the correlation coefficient ρ_{XY} between X and Y.

d. (5) Find the density function $f_Y(y)$ for Y.

(a) $E(X) = \overset{+1}{\int_{-\infty}^{\infty} x\, f_X(x)\, dx} = \overset{+1}{\int_{-\infty}^{\infty} x|x|\, dx} = \int_{-1}^{0} -x^2\, dx + \int_0^1 x^2\, dx = \overset{+1}{0}$

$Var(X) = \overset{+1}{\int_{-\infty}^{\infty} (x - E(x))^2 f_X(x)} = \overset{+1}{\int_{-\infty}^{\infty} x^2 |x|\, dx} = \int_{-1}^{0} -x^3\, dx + \int_0^1 x^3\, dx = \overset{+1}{\frac{1}{2}}$

or

$Var(X) = \underset{+1}{E[X^2] - (E[X])^2} = \frac{1}{2} - 0^2 = \frac{1}{2}$ +2

(b) $E(Y) = E(|X|) = \overset{+1}{\int_{-\infty}^{\infty} |x| f_X(x)\, dx} = \overset{+1}{\int_{-\infty}^{\infty} |x||x|\, dx}$

$= \int_{-1}^{0} x^2\, dx + \int_0^1 x^2\, dx = \overset{+1}{\frac{2}{3}}$

$Var(Y) = Var(|X|) = \overset{+2}{\int_{-\infty}^{\infty} (|x| - E(|x|))^2 f_X(x)\, dx} = \int_{-1}^{1} (|x| - \tfrac{2}{3})^2 |x|\, dx = \overset{+1}{\frac{1}{18}}$

or

$Var(Y) = \overset{+2}{E(Y^2) - (E(Y))^2} = \frac{1}{2} - (\tfrac{2}{3})^2 = \overset{+1}{\frac{1}{18}}$

3. (continued – 1)

(c) $E[XY] = E[X|X|] = \int_{-\infty}^{\infty} x|x| f_X(x)\,dx \overset{+2}{=} \int_{-1}^{1} x|x|^2\,dx$ +1

$= \int_{-1}^{0} x^3\,dx + \int_{0}^{1} x^3\,dx = 0$ +1

$\rho_{XY} = \dfrac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} \overset{+2}{=} \dfrac{0 - 0 \cdot \frac{2}{3}}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{18}}} = 0$ +1

(d) Method 1 :

$F_Y(y) = P(Y \le y) = P(|X| \le y) \overset{+2}{=} P(-y \le x \le y)$

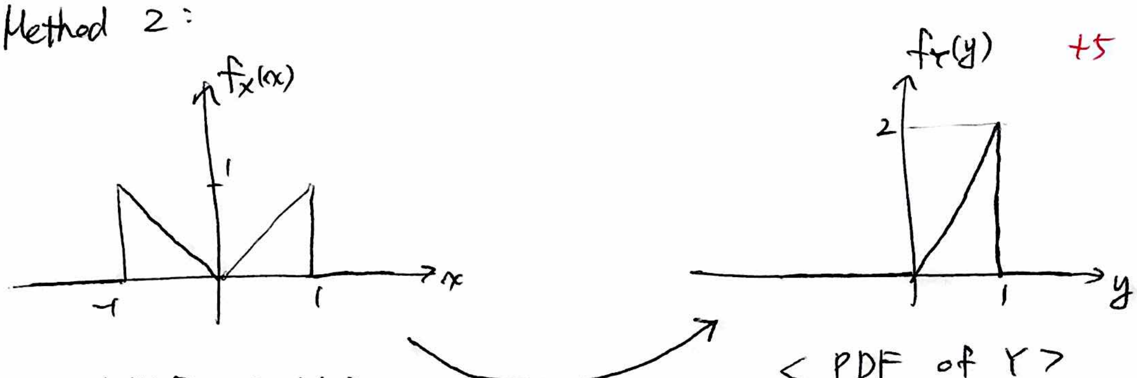
$= \int_{-y}^{y} f_X(x)\,dx = \int_{-y}^{y} |x|\,dx \ , \quad -1 \le x \le 1$

$= \int_{-y}^{0} (-x)\,dx + \int_{0}^{y} x\,dx$

$= \frac{1}{2}y^2 + \frac{1}{2}y^2 = y^2 \ , \quad 0 \le y \le 1$ +2

$f_Y(y) = \dfrac{dF_Y(y)}{dy} = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$ +1

Method 2 :



< PDF of X > < PDF of Y >

Since $Y = |X|$, the left triangle is flipped along the y-axis. So, the two triangles merge up to make one taller triangle at the right side.

4. (25 pts) Consider the 2-D signal $f(x,y)=1+\cos\big(2\pi(90x+10y)\big)$. Here the units of x and y are inches.

   a. (7) Sketch what this signal would look like. Be sure to dimension all important quantities. (Your sketch doesn't need to be a piece of artwork. You only need to convince us that you know what the signal looks like.)

   We sample $f(x,y)$ with an ideal sampler at 100 samples/inch to generate a new signal $f_s(x,y)=\text{comb}_{0.01,\,0.01}\big[f(x,y)\big]$.

   b. (9) Find a simple expression for the CSFT $F_s(u,v)$ of $f_s(x,y)$. Your answer should not contain any operators. Sketch what the spectrum $F_s(u,v)$ would look like. Be sure to dimension all important quantities.

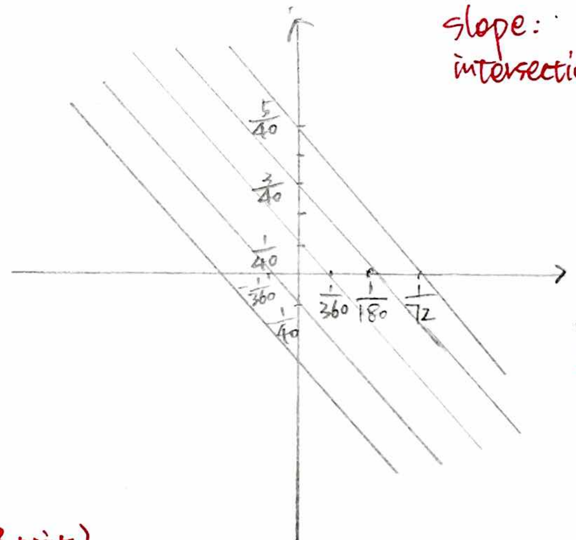
   We reconstruct a continuous-space version of $f_s(x,y)$ by convolving $f_s(x,y)$ with the function $h_r(x,y)=\text{sinc}\big(100x,100y\big)$ to generate the reconstructed signal $f_r(x,y)$.

   c. (9) Find a simple expression for $f_r(x,y)$. Your answer should not contain any operators. Sketch what $f_r(x,y)$ would look like. Be sure to dimension all important quantities.

a) $f(x,y)=1+\cos(2\pi(90x+10y))$

$2\pi(90x+10y)=\frac{\pi}{2}+k\pi$

$y=-9x+\frac{2k+1}{40}$ 3 points



slope:
intersection: 2 points.

parallel 2 points.

b) $F(u,v)=\delta(u,v)+\frac{1}{2}\delta(u-90,v-10)$
$+\frac{1}{2}\delta(u+90,v+10)$ (2 points)

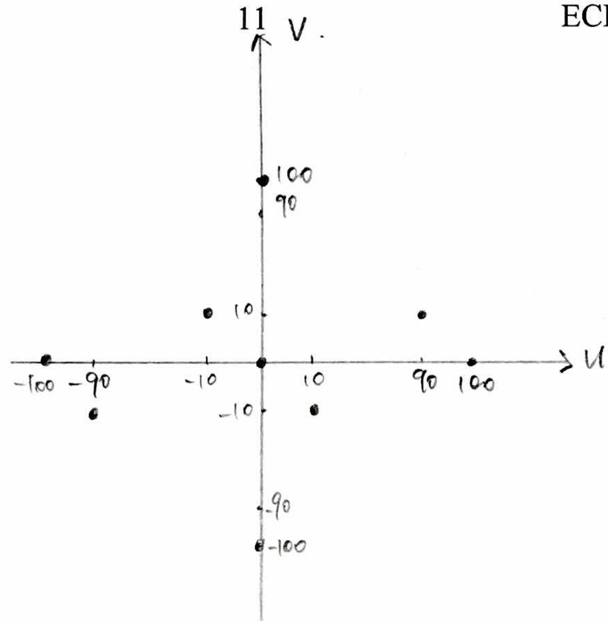
$F_s(u,v)=100^2\,\text{rep}_{100,100}[(u,v)]$ ~~(3 points)~~.

$=10000\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\Big[\delta(u-100k,v-100l)+\frac{1}{2}\big[\delta(u-100k-90,v-100l-10)$
$+\delta(u-100k+90,v-100l+10)\big]\Big]$ (3 points).

4. (continued – 1)

4 points

b)



c) $h_r(x,y) = \mathrm{sinc}(100x, 100y)$

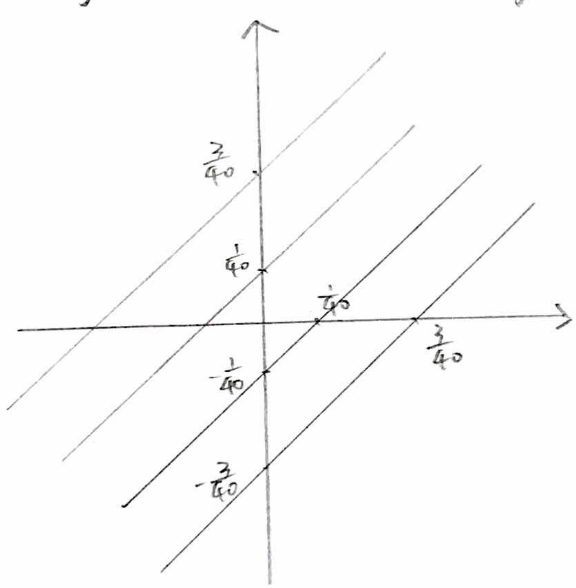
2 points.

$H_r(u,v) = \frac{1}{100^2} \cdot \mathrm{rect}\left(\frac{x}{100}, \frac{y}{100}\right)$

$F_r(u,v) = F_s(u,v)\, H_r(u,v)$

$= \delta(u,v) + \frac{1}{2}\delta(u-10, v+10) + \frac{1}{2}\delta(u+10, v-10)$ 3 points.

$\Rightarrow f_r(x,y) = 1 + \cos(2\pi(10x - 10y))$



parallel.

4 points.

5. (25 pts) Consider a spatial filter with *separable* point spread function $h[m,n]$ given below

	$h[m,n]$	$n = -1$	$n = 0$	$n = 1$
m	-1	$-1/8$	$-1/4$	$-1/8$
	0	$1/2$	1	$1/2$
	1	$-1/8$	$-1/4$	$-1/8$

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 11×11 set of pixels in the input image.

0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, the $\mu = \nu$ axis, and the $\mu = -\nu$ axis.

c. (3) Using your results from parts a) and b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

5. (continued – 1)

a)

$$
\begin{matrix}
-1/8 & -3/8 & -1/2 & -1/2 & -1/2 & -3/8 & -1/8 & 0 & 0 & 0 & 0 \\
3/8 & 9/8 & 3/2 & 3/2 & 3/2 & 1 & 1/8 & -1/8 & 0 & 0 & 0 \\
1/4 & 3/4 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 & 0 & 0 \\
1/4 & 3/4 & 1 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 & 0 \\
1/4 & 3/4 & 1 & 1 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 \\
1/4 & 3/4 & 1 & 1 & 1 & 1 & 1 & 1 & 5/4 & 5/4 & 1/2 \\
1/4 & 3/4 & 1 & 1 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 \\
1/4 & 3/4 & 1 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 & 0 \\
1/4 & 3/4 & 1 & 1 & 1 & 9/8 & 7/8 & 1/8 & -1/8 & 0 & 0 \\
3/8 & 9/8 & 3/2 & 3/2 & 3/2 & 1 & 1/8 & -1/8 & 0 & 0 & 0 \\
-1/8 & -3/8 & -1/2 & -1/2 & -1/2 & -3/8 & -1/8 & 0 & 0 & 0 & 0
\end{matrix}
$$

b) $h[m,n] = \frac{1}{8} h_1[m] h_2[m]$

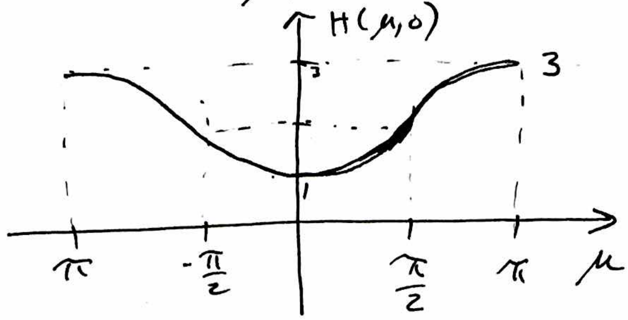
where $h_1[m] = \{-1, \underset{\substack{\uparrow \\ m=0}}{\underline{4}}, -1\}$

$h_2[n] = \{1 \ \underset{\substack{\uparrow \\ n=0}}{\underline{2}} \ 1\}$

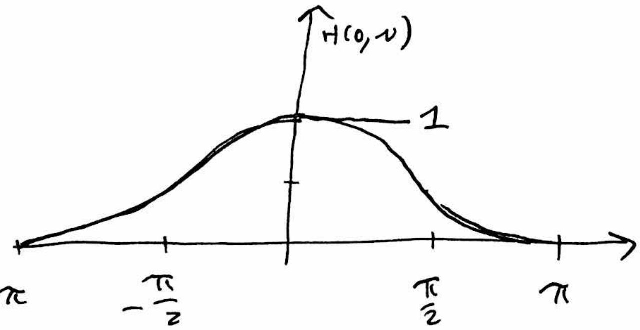
Then
$$
\begin{aligned}
H(\mu,\nu) &= \frac{1}{8} H_1(\mu) H_2(\nu) \\
&= \frac{1}{8}\left(-e^{j\mu} + 4 - e^{-j\mu}\right)\left(e^{j\nu} + 2 + e^{-j\nu}\right) \\
&= \frac{1}{8}\left(4 - 2\cos\mu\right)\left(2 + 2\cos\nu\right) \\
&= \frac{1}{2}\left(2 - \cos\mu\right)\left(1 + \cos\nu\right)
\end{aligned}
$$

5. (continued – 2)

b) $H(\mu, 0) = \frac{1}{2}(2-\cos\mu)(1+1) = 2-\cos\mu$



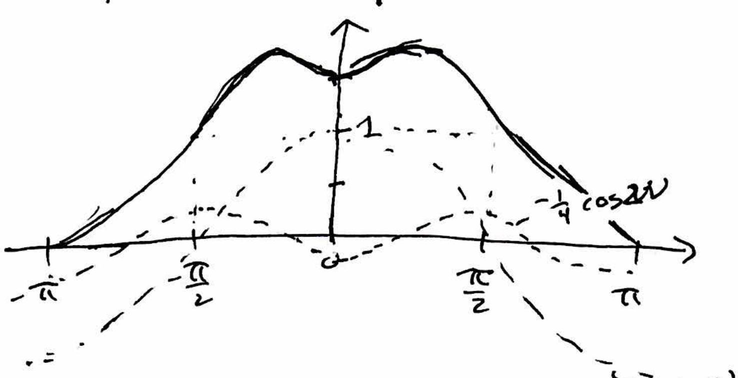
$H(0,\nu) = \frac{1}{2}(2-1)(1+\cos\nu) = \frac{1}{2}(1+\cos\nu)$



$H(\mu=\nu,\nu) = \frac{1}{2}(2-\cos\nu)(1+\cos\nu) = \frac{1}{2}(2-\cos\nu+2\cos\nu-\cos^2\nu)$

$= \frac{1}{2}\left(2+\cos\nu-\frac{1}{2} - \frac{1}{2}\cos 2\nu\right)$

$= \frac{3}{4} + \cos\nu - \frac{1}{4}\cos 2\nu$



$H(\mu=-\nu,\nu) = \frac{1}{2}(2-\cos(-\nu))(1+\cos\nu) = \frac{1}{2}(2-\cos\nu)(1+\cos\nu) = H(\nu,\nu)$

Same as above

c) From $H(\mu,0)$, we see that the filter is ~~low~~ high pass in the vertical direction, along m, and low pass in the horizontal direction, along n. Edges are intensified along m and smooth across n. Also, the center stays 1 because $\sum\sum k[m,n] = 1$ (the DC level is preserved).