20 April 2016 **Name:**____________________

ECE 438 **Exam No. 3** **Spring 2016**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a random variable X with density function

$$f_X(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & \text{else} \end{cases}. \qquad -1 < x < 1$$

a. (9) Find the mean and variance of X. 0 $\frac{1}{2}$

Suppose we generate a new random variable $Y = Q(X)$ by quantizing X according to the following quantizer:

$$Q(x) = \begin{cases} \frac{2}{3}, & \frac{1}{3} \le x \\ 0, & -\frac{1}{3} \le x < \frac{1}{3} \\ -\frac{2}{3}, & x < -\frac{1}{3} \end{cases}$$

b. (8) Find the mean and variance of Y. 0 $\frac{32}{81}$

c. (8) Determine the correlation coefficient ρ_{XY} between X and Y. $\frac{26}{27}$

a) $E(x) = \int_{-\infty}^{\infty} x f_x(x)\,dx$ 1

$= \int_{-1}^{0} x(-x)\,dx + \int_{0}^{1} x(x)\,dx$ 1

$= -\frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_{0}^{1}$ 1

$= -\frac{1}{3} + \frac{1}{3} = 0$ (3 Points)

$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx$ 1

$= \int_{-1}^{0} x^2(-x)\,dx + \int_{0}^{1} x^2(x)\,dx$ 1

$= -\frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{0}^{1}$ 1

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ (3 Points)

$Var(x) = E(x^2) - (E(x))^2 = \frac{1}{2}$

(3 Points)

$\frac{1}{2}$ 1

1. (continued - 1)

b) $E(Y) = \int_{-\infty}^{\infty} Q(x) f_X(x)\, dx$ 1

$= \int_{-1}^{-\frac{1}{3}} \left(-\frac{2}{3}\right)(-x)\, dx + \int_{\frac{1}{3}}^{1} \frac{2}{3} x\, dx$ 1

$= \frac{x^2}{3}\Big|_{-1}^{-\frac{1}{3}} + \frac{x^2}{3}\Big|_{\frac{1}{3}}^{1}$ 1

$= \left(\frac{1}{9} - 1\right)/3 + \left(1 - \frac{1}{9}\right)/3 = 0$ (3 Points)

$E(Y^2) = \int_{-\infty}^{\infty} Q^2(x) f_X(x)\, dx$ 1

$= \int_{-1}^{-\frac{1}{3}} \left(\frac{4}{9}\right)(-x)\, dx + \int_{\frac{1}{3}}^{1} \frac{4}{9} x\, dx$ 1

$= -\frac{2}{9} x^2 \Big|_{-1}^{-\frac{1}{3}} + \frac{2}{9} x^2 \Big|_{\frac{1}{3}}^{1}$ 1

$= \left(\frac{1}{9} - 1\right) \times \left(-\frac{2}{9}\right) + \left(1 - \frac{1}{9}\right) \times \frac{2}{9} = \frac{32}{81}$ (3 Points)

$Var(Y) = E(Y^2) - \left(E(Y)\right)^2 = \frac{32}{81}$ (2 Points)

1 1

c). $\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$ (2 Points)

$\sigma_{XY}^2 = E(XY) - E(X)E(Y)$ (2 Points)

$E(XY) = \int_{-\infty}^{\infty} x\, Q(x) f_X(x)\, dx$ 1

$= \int_{-1}^{-\frac{1}{3}} x \cdot \left(-\frac{2}{3}\right) \cdot (-x) + \int_{\frac{1}{3}}^{1} x\, \frac{2}{3} \cdot x\, dx$

$= \frac{2}{9} x^3 \Big|_{-1}^{-\frac{1}{3}} + \frac{2}{9} x^3 \Big|_{\frac{1}{3}}^{1}$ 1

$= \frac{2}{9} \cdot \left(-\frac{1}{27} + 1\right) + \frac{2}{9} \cdot \left(1 - \frac{1}{27}\right) = \frac{2}{9} \times 2 \times \frac{26}{27} = \frac{4 \cdot 26}{9 \cdot 27}$ (2 Points)

$\rho_{XY} = \frac{\frac{4 \cdot 26}{9 \cdot 27}}{\sqrt{\frac{1}{2} \cdot \frac{32}{81}}} = \frac{\frac{4 \cdot 26}{9 \cdot 27}}{\sqrt{\frac{16}{81}}} = \frac{\cancel{4}}{\cancel{9}} \cdot \frac{26}{27} \cdot \frac{\cancel{9}}{\cancel{4}} = \frac{26}{27}$ (2 Points)

2. (25) Let $X[n]$ be a wide-sense stationary sequence of random variables with zero mean and autocorrelation function

$$r_{XX}[n] = \begin{cases} 1, & n=0 \\ \frac{1}{2}, & |n|=1 \\ 0, & \text{else} \end{cases}$$

Suppose that this sequence is processed with the following filter to generate the output sequence $Y[n]$:

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$$

a. (3) Find the mean of the sequence $Y[n]$.

b. (12) Find the cross-correlation $r_{XY}[n]$ between X and Y.

c. (10) Find the autocorrelation $r_{YY}[n]$ of the output Y.

(a) $E\{Y[n]\} = E\{\frac{1}{2}(x[n] + x[n-1])\} = \frac{1}{2}E\{x[n]\} + \frac{1}{2}E\{x[n-1]\}$

since $x[n]$ is wide-sense stationary,

$E\{x[n]\} = E\{x[n-1]\} = 0 \Rightarrow E\{Y[n]\} = 0$

(b) $h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1] = \{\underset{\uparrow n=0}{\frac{1}{2}}, \frac{1}{2}\}$

$r_{xx}[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1] = \{\frac{1}{2}, \underset{\uparrow n=0}{1}, \frac{1}{2}\}$

$r_{xy}[n] = h[n] * r_{xx}[n] =$ option 1 : $= \frac{1}{2}r_{xx}[n] + \frac{1}{2}r_{xx}[n-1] = \{\frac{1}{4}, \underset{\uparrow n=0}{\frac{1}{2}}, \frac{1}{4}\}$

$+ \{\underset{\uparrow}{\frac{1}{4}}, \frac{1}{2}, \frac{1}{4}\}$

$= \{\frac{1}{4}, \underset{\uparrow}{\frac{3}{4}}, \frac{3}{4}, \frac{1}{4}\}$

option 2 : $= \{\underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}\} * \{\frac{1}{2}, \underset{\uparrow}{1}, \frac{1}{2}\} = \{\frac{1}{4}, \underset{\uparrow}{\frac{3}{4}}, \frac{3}{4}, \frac{1}{4}\}$

option 3 : $= \frac{1}{2}\left(\frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]\right)$

$+ \frac{1}{2}\left(\frac{1}{2}\delta[n] + \delta[n-1] + \frac{1}{2}\delta[n-2]\right)$

$= \frac{1}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{3}{4}\delta[n-1] + \frac{1}{4}\delta[n-2]$

2. (continued - 1)

(c) $r_{YY}[n] = h[n] * r_{xy}[-n]$

if wrong, (-3) =

option 1 : $\{\overset{\downarrow}{\tfrac{1}{2}}, \tfrac{1}{2}\} * \{\tfrac{1}{4}, \tfrac{3}{4}, \overset{\downarrow}{\tfrac{3}{4}}, \tfrac{1}{4}\} = \{\tfrac{1}{8}, \tfrac{1}{2}, \overset{\downarrow}{\tfrac{3}{4}}, \tfrac{1}{2}, \tfrac{1}{8}\}$

option 2 : $\frac{1}{2} r_{xy}[-n] + \frac{1}{2} r_{xy}[-(n-1)]$

$$= \frac{1}{2}\{\tfrac{1}{4}, \tfrac{3}{4}, \overset{\downarrow}{\tfrac{3}{4}}, \tfrac{1}{4}\} + \frac{1}{2}\{\tfrac{1}{4}, \overset{\downarrow}{\tfrac{3}{4}}, \tfrac{3}{4}, \tfrac{1}{4}\}$$

$$= \frac{1}{2}\{\tfrac{1}{4}, 1, \overset{\downarrow}{\tfrac{3}{2}}, 1, \tfrac{1}{4}\} = \{\tfrac{1}{8}, \tfrac{1}{2}, \overset{\downarrow}{\tfrac{3}{4}}, \tfrac{1}{2}, \tfrac{1}{8}\}$$

option 3 : $\left(\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]\right) * \left(\frac{1}{4}\delta[n+2] + \frac{3}{4}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{1}{4}\delta[n-1]\right)$

$$= \frac{1}{8}\delta[n+2] + \frac{3}{8}\delta[n+1] + \frac{3}{8}\delta[n] + \frac{1}{8}\delta[n-1]$$
$$+ \frac{1}{8}\delta[n+1] + \frac{3}{8}\delta[n] + \frac{3}{8}\delta[n-1] + \frac{1}{8}\delta[n-2]$$

$$= \frac{1}{8}\delta[n+2] + \frac{1}{2}\delta[n+1] + \frac{3}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{8}\delta[n-2]$$

3. (25) Consider the continuous-time signal

$$x(t) = \text{rep}_{1/100}\left[rect(t/(1/200))\right]\cos\left(2\pi(800)t\right)$$

a. (4) Sketch $x(t)$. Be sure to label all-important quantities. Assume that the units of the time variable t is seconds.

b. (5) Using transform relations, find a simple expression for the CTFT $X(f)$ of $x(t)$.

c. (4) Sketch $X(f)$. Be sure to label all-important quantities.

d. (4) Suppose that $x(t)$ is a model for a speech waveform for a voiced epoch of speech. What would be the vocal tact response, pitch period, and formant frequency?

Define the short-time, continuous-time Fourier transform of $x(t)$ as

$$\tilde{X}(f,t) = \int_{-\infty}^{\infty} x(\tau)w(t-\tau)e^{-j2\pi f\tau}d\tau\ ,$$

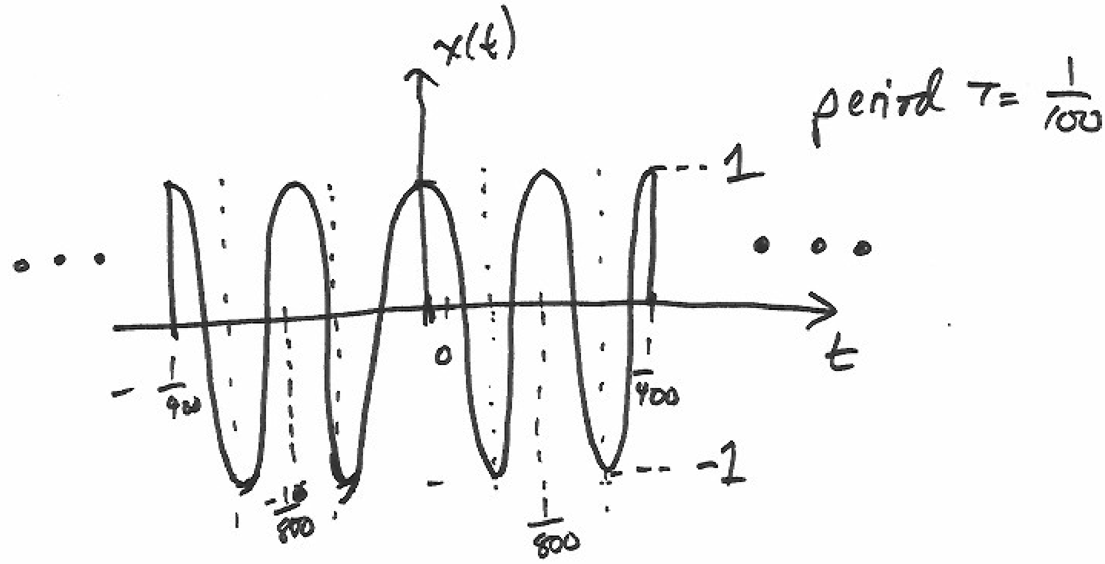
where for our particular problem, the window is defined to be

$$w(\tau) = rect(\tau/(1/200)\ .$$

e. (4) Find a simple expression for $\tilde{X}(f,0)$.

f. (4) Find a simple expression for $\tilde{X}(f,1/200)$.

a)



3. (continued - 1)

b)

$$\text{rep}_T\{x(t)\} \xleftrightarrow{CTFT} \frac{1}{T}\,\text{comb}_{\frac{1}{T}}\{X(f)\}$$

$$\text{rect}(t) \xleftrightarrow{CTFT} \text{sinc}(f)$$

$$x\left(\frac{t}{a}\right) \xleftrightarrow{CTFT} |a|\,X(af)$$

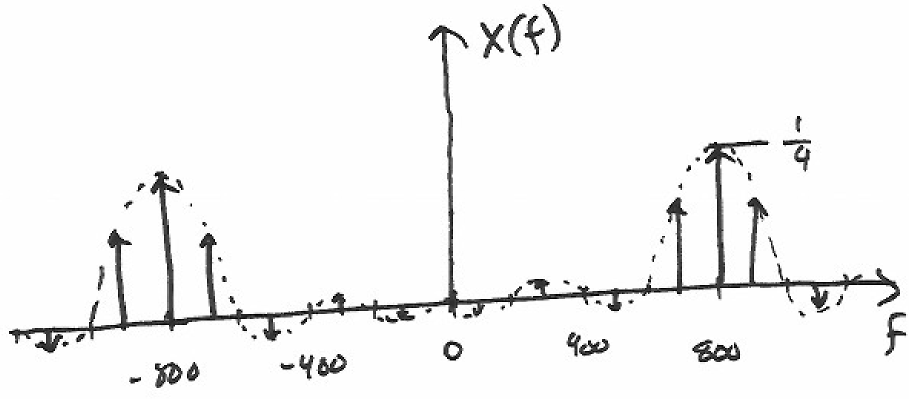
$$\cos(2\pi f_0 t) \xleftrightarrow{CTFT} \frac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)$$

Then $$x(t) \xleftrightarrow{CTFT} 100\,\text{comb}_{100}\left\{\frac{1}{200}\,\text{sinc}\left(\frac{f}{200}\right)\right\} * \left(\frac{1}{2}\left(\delta(f-800)+\delta(f+800)\right)\right)$$

$$= \frac{1}{4}\,\text{comb}_{100}\left\{\text{sinc}\left(\frac{f-800}{200}\right)\right\} + \frac{1}{4}\,\text{comb}_{100}\left\{\text{sinc}\left(\frac{f+800}{200}\right)\right\}$$

$$= \frac{1}{4}\,\text{comb}_{100}\left\{\text{sinc}\left(\frac{f-800}{200}\right) + \text{sinc}\left(\frac{f+800}{200}\right)\right\}$$

c)



d)

Vocal tract response: $v(t) = \text{rect}\left(t/(1/200)\right)\cos(2\pi 800 t)$

pitch period: $T = \frac{1}{100}$ s

Formant frequency: $F = 800$ Hz

3. (continued - 2)

e) For parts e) and f), notice that $\tilde{X}(f,t)$ is the CTFT of a windowed portion of $x(t)$.

For $\tilde{X}(f,0)$, the window is directly overlaying one pulse,

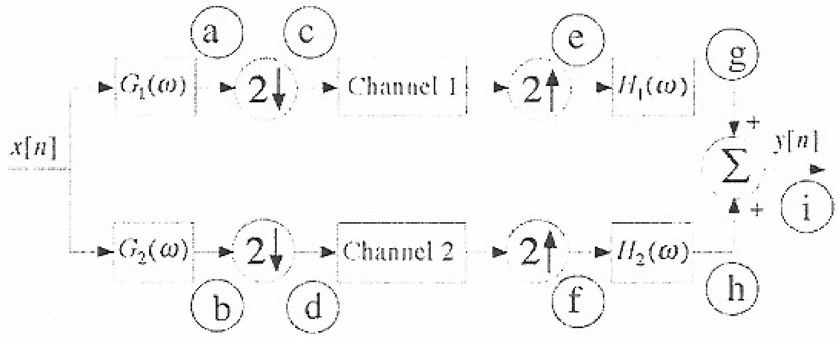
therefore $\tilde{X}(f,0) = \text{CTFT}\left\{\text{rect}\left(t/(1/200)\right)\cos(2\pi 800 t)\right\}$

$$= \left(\frac{1}{200}\,\text{sinc}\left(\frac{f}{200}\right)\right) * \frac{1}{2}\left(\delta(f-800) + \delta(f+800)\right)$$

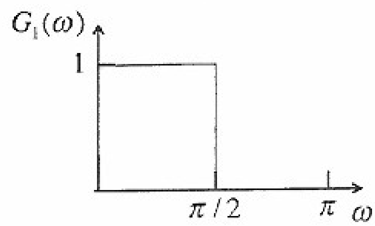
$$= \frac{1}{400}\left(\text{sinc}\left(\frac{f-800}{200}\right) + \text{sinc}\left(\frac{f+800}{200}\right)\right)$$

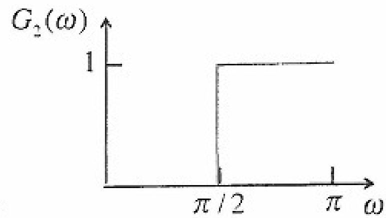
f) when ~~we~~ we have $\tilde{X}(f, \frac{1}{200})$, the window is overlaying $x(t)$ where $x(t)$ is zero, therefore $\tilde{X}(f, \frac{1}{200}) = 0$.

4. (25) Consider the 2-channel filter bank shown below



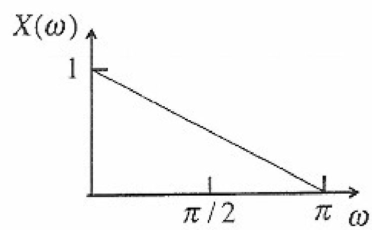
Assume that $H_1(\omega) = G_1(\omega)$, $H_2(\omega) = G_2(\omega)$, and these filters have frequency responses as shown below:





Here we assume that $G_1(\omega) = G_1(-\omega)$ and $G_2(\omega) = G_2(-\omega)$, and of course, the frequency responses are periodic in ω with period 2π. Assume also for purposes of this problem, that Channels 1 and 2 do nothing, i.e. they are identity functions.

Suppose that the DTFT $X(\omega)$ of the input $x[n]$ to the filter bank is given by:

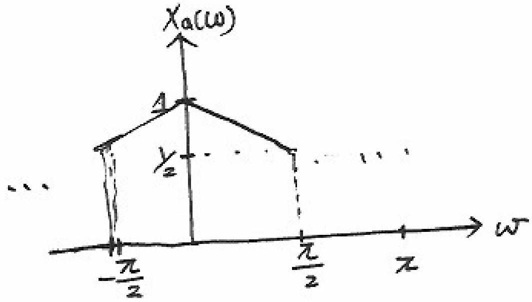


Define the DTFT of the signal at points (a)-(i) in the filter bank diagram above as $X_k(\omega)$, where $k =$ a, b, c, ..., i. Note that $X_i(\omega) = Y(\omega)$.

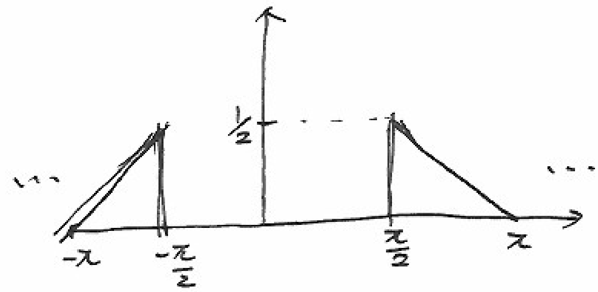
Sketch $X_k(\omega)$, for all nine cases, where $k =$ a, b, c, ..., i. Be sure to label all important dimensions in your sketches. (The plots are worth the following: (a) and (b) – 5 pts, (c) and (d) – 5 pts, (e) and (f) – 5 pts, (g) and (h) – 5 pts, (i) – 5 pts.)

4. (continued - 1)

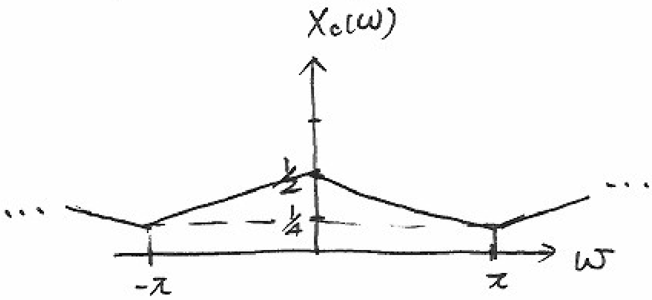
a) $X_a(\omega) = X(\omega)G_1(\omega)$



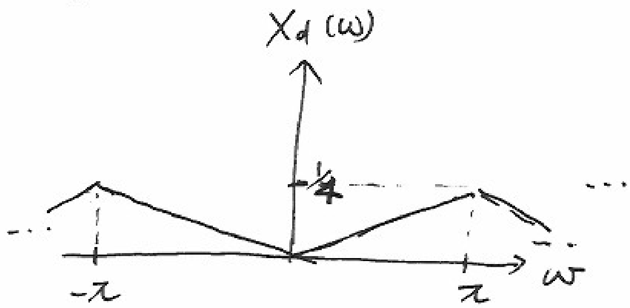
b) $X_b(\omega) = X(\omega)G_2(\omega)$



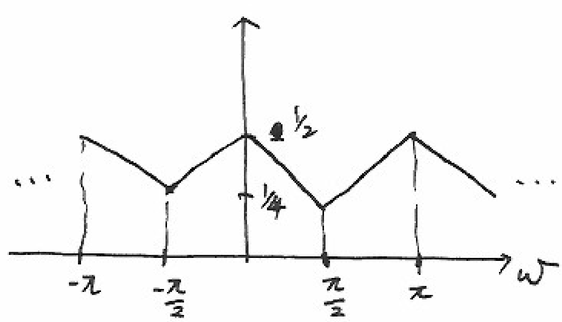
(c) $X_c(\omega) = \frac{1}{2}\sum_{k=0}^{1} X_a\left(\frac{\omega-2\pi k}{2}\right)$



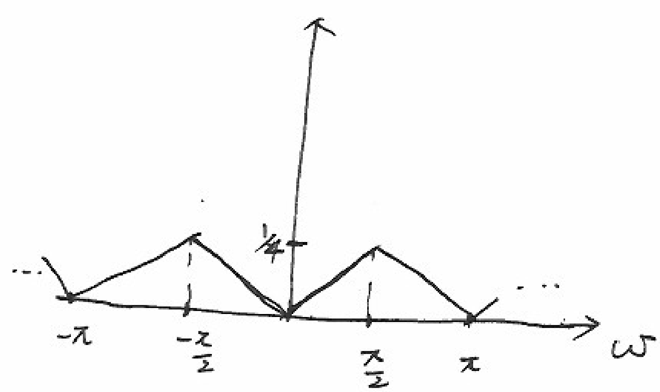
(d) $X_d(\omega) = \frac{1}{2}\sum_{k=0}^{1} X_b\left(\frac{\omega-2\pi k}{2}\right)$



(e) $X_e(\omega) = X_c(2\omega)$

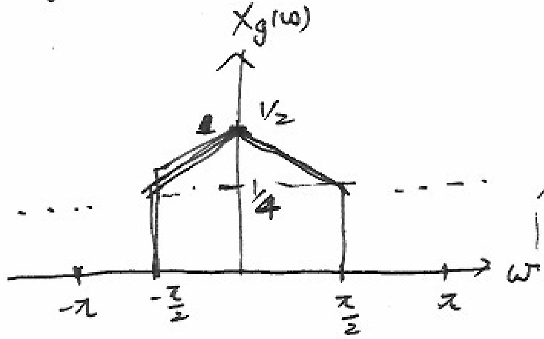


(f) $X_f(\omega) = X_d(\omega)$

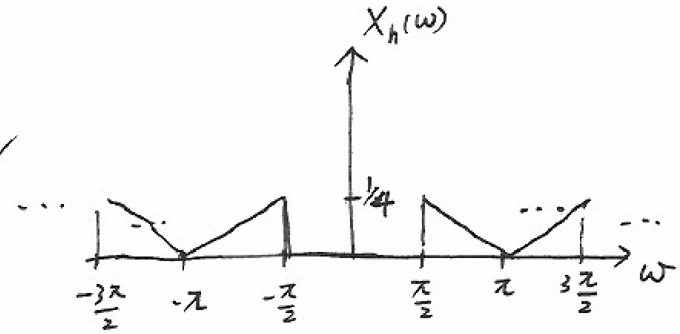


4. (continued - 2)

(g) $X_g(\omega) = X_e(\omega) H_1(\omega)$



h) $X_h(\omega) = X_f(\omega) H_2(\omega)$



(i) $X_i(\omega) = X_g(\omega) + X_h(\omega)$