23 March 2016

Name: SOLUTION

ECE 438 **Exam No. 2** **Spring 2016**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a causal LTI system described by the difference equation $y[n]=x[n]-\frac{2}{3}y[n-1]$ and input $x[n]=\left(\frac{1}{3}\right)^n u[n]$.

   a. (10) Find the Z-transform $Y(z)$ of the output $y[n]$, and state the region of convergence.

   b. (15) Find the inverse ZT for $Y(z)$ to determine $y[n]$.

a) $y[n]=x[n]-\frac{2}{3}y[n-1]$

Using $x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)$

$$Y(z)=X(z)-\frac{2}{3}z^{-1}Y(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1+\frac{2}{3}z^{-1}}$$ ROC: $|z|>\frac{2}{3}$ because it is a causal system

$$X(z)=\frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC}: |z|>\frac{1}{3}$$

using the table

$$\Rightarrow Y(z)=H(z)X(z)=\frac{1}{(1+\frac{2}{3}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$\text{ROC}\{Y(z)\}=\text{ROC}\{H(z)\}\cap\text{ROC}\{X(z)\} \Rightarrow |z|>\frac{2}{3}$$

b) $$Y(z)=\frac{A}{1+\frac{2}{3}z^{-1}}+\frac{B}{1-\frac{1}{3}z^{-1}}$$

$$A=\left.\frac{1}{1-\frac{1}{3}z^{-1}}\right|_{z^{-1}=-\frac{3}{2}}=\frac{1}{1+\frac{1}{3}\cdot\frac{3}{2}}=\frac{2}{3}$$

$$B=\left.\frac{1}{1+\frac{2}{3}z^{-1}}\right|_{z^{-1}=3}=\frac{1}{1+\frac{2}{3}\cdot 3}=\frac{1}{3}$$

1. (continued)

$$Y(z) = \frac{2}{3}\frac{1}{1+\frac{2}{3}z^{-1}} + \frac{1}{3}\frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{2}{3}$$

$$y[n] = \frac{2}{3}\left(-\frac{2}{3}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{3}\right)^n u[n]$$

2. (25) Fast Fourier Transform Algorithm

   a. (13) Derive the complete equations that describe a Fast Fourier Transform (FFT) Algorithm to compute a 15-point Discrete Fourier Transform (DFT).

   b. (12) Draw a complete and fully labeled flow diagram for your 15-point FFT algorithm.

a)

15-Pt DFT = 3×5-Pt DFT (2 Points)

$$X^{(15)}[k] = \sum_{n=0}^{14} x[n]\, e^{-j\frac{2\pi k n}{15}}, \quad k = 0, \dots 14$$ (3 Points).

$$= \sum_{l=0}^{4} \sum_{m=0}^{2} x[3l+m]\, e^{-j\frac{2\pi k}{15}\cdot(3l+m)}$$ (3 Points).

$$= \sum_{l=0}^{4} x[3l]\, e^{-j\frac{2\pi k}{15}\cdot 3l} + \sum_{l=0}^{4} x[3l+1]\, e^{-j\frac{2\pi k}{15}\cdot 3l} \cdot e^{-j\frac{2\pi k}{15}}$$

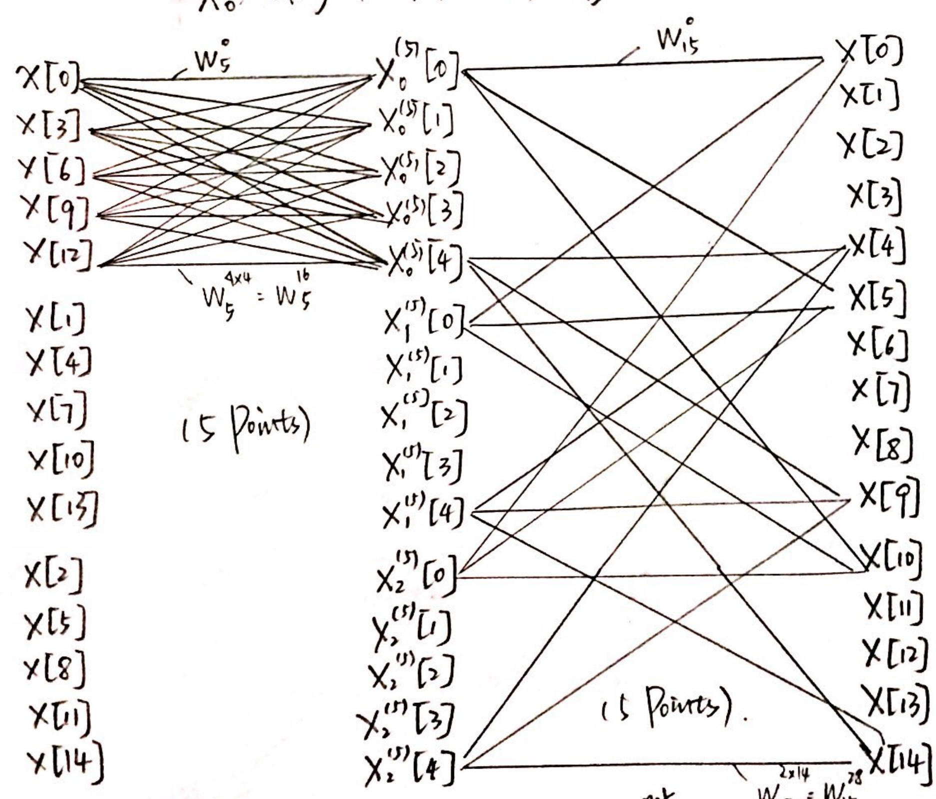
$$+ \sum_{l=0}^{4} x[3l+2]\, e^{-j\frac{2\pi k}{15}\cdot 3l} \cdot e^{-j\frac{2\pi k}{15}\cdot 2}$$

$$= X_0^{(5)}(k') + X_1^{(5)}(k')\, W_{15}^{k} + X_2^{(5)}(k')\, W_{15}^{2k}$$

$W_N^k = e^{-j\frac{2\pi k}{N}}$ (5 Points)

$k' = k \bmod 5$

b)



$W_5^{lk} = e^{-j\frac{2\pi l k}{5}}$ (1 Points)

$W_{15}^{mk} = e^{-j\frac{2\pi m k}{15}}$ (1 Points).

3. (25 pts) DFT properties. Let $x[n], n=0,...,N-1$ be an N-point signal with N-point DFT $X^{(N)}[k], k=0,...,N-1$.

a) (13) Define a new N-point signal $y[n]=(-1)^n x[n], n=0,...,N-1$. Find a simple expression for the N-point DFT $Y^{(N)}[k], k=0,...,N-1$ in terms of $X^{(N)}[k], k=0,...,N-1$.

b) (12) Define a new N-point signal $y[n]=x[n]+x[(N-1)-n], n=0,...,N-1$. Find a simple expression for the N-point DFT $Y^{(N)}[k], k=0,...,N-1$ in terms of $X^{(N)}[k], k=0,...,N-1$.

Solution for part (a)

$$Y^{(N)}[k]=\sum_{n=0}^{N-1}(-1)^n x[n]e^{-j2\pi kn/N}$$

$$=\sum_{n=0}^{N-1}(e^{j\pi})^n x[n]e^{-j2\pi kn/N}$$

$$=\sum_{n=0}^{N-1}e^{j\pi n}x[n]e^{-j2\pi kn/N}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j2\pi(k-N/2)n/N}$$

$$=X^{(N)}[k-N/2],\ k=0,...,N-1$$

(Here we assume that N is even. This answer is sufficient for full credit)

$$=\begin{cases} X^{(N)}[k+N/2], & k=0,...,N/2-1 \\ X^{(N)}[k-N/2], & k=N/2,...,N-1 \end{cases}$$

(Here we have used the fact that $X^{(N)}[k]$ is periodic with period N.)

Solution for part (b)

$$Y^{(N)}[k]=\sum_{n=0}^{N-1}\left(x[n]+x[(N-1)-n]\right)e^{-j2\pi kn/N}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}+\sum_{n=0}^{N-1}x[(N-1)-n]e^{-j2\pi kn/N}$$

$$=X^{(N)}[k]+\sum_{m=N-1}^{0}x[m]e^{-j2\pi k((N-1)-m)/N}$$

where we have let $m=(N-1)-n \Rightarrow n=(N-1)-m$ in the second summation

$$=X^{(N)}[k]+\sum_{m=N-1}^{0}x[m]e^{-j2\pi k(-m)/N}e^{-j2\pi k(N-1)/N}$$

$$=X^{(N)}[k]+e^{-j2\pi k(N-1)/N}\sum_{m=0}^{N-1}x[m]e^{-j2\pi(-k)m/N}$$

$$=X^{(N)}[k]+e^{-j2\pi k(N-1)/N}X^{(N)}[-k]$$

Note from Prof. Allebach: I should have stated this part of the problem as

$$y[n] = x[n] + x[N-n]$$

Then we would have:

$$\begin{aligned} Y^{(N)}[k] &= \sum_{n=0}^{N-1} \left(x[n] + x[N-n]\right) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[N-n] e^{-j2\pi kn/N} \\ &= X^{(N)}[k] + \sum_{m=N-1}^{0} x[m] e^{-j2\pi k(N-m)/N} \end{aligned}$$

where we have let $m = N - n \Rightarrow n = N - m$ in the second summation

$$\begin{aligned} &= X^{(N)}[k] + \sum_{m=N}^{1} x[m] e^{-j2\pi k(-m)/N} e^{-j2\pi kN/N} \\ &= X^{(N)}[k] + \sum_{m=N}^{1} x[m] e^{-j2\pi k(-m)/N} \\ &= X^{(N)}[k] + \sum_{m=0}^{N-1} x[m] e^{-j2\pi(-k)m/N} \end{aligned}$$

Here we have taken advantage of the fact that $x[0] = x[N]$ and $e^{-j2\pi(-k)(0)/N} = e^{-j2\pi(-k)(N)/N}$

So we get:

$$Y^{(N)}[k] = X^{(N)}[k] + X^{(N)}[-k]$$

Note that if $x[n]$ is real-valued, we have that $X^{(N)}[-k] = \left(X^{(N)}[k]\right)^*$ and

$$\begin{aligned} Y^{(N)}[k] &= 2\,\mathrm{Re}\left\{X^{(N)}[k]\right\} \\ &= 2\sum_{n=0}^{N-1} x[n] \cos(2\pi kn / N) \end{aligned}$$

So $y[n]$ and $Y^{(N)}[k]$ are both real and even.

4. (25 pts) Spectral analysis with the DFT. Consider the 10-point signal $x[n]=\cos(5\pi n/10), n=0,...,9$.

a) (13) Find a simple closed-form expression for the 10-point DFT $X^{(10)}[k]$ of $x[n]$ in terms of the function $\text{psinc}_N(\omega)=\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$.

b) (12) Carefully draw a complete sketch of $X^{(10)}[k], k=0,...,9$ being sure to dimension all important quantities and fully label the axes. You may ignore the contribution of any linear phase terms to your sketch.

a) Let $y[n]=\cos\left(\frac{5\pi}{10}n\right)$ and $w[n]=u[n]-u[n-10]$

then $x[n]=y[n]\,w[n]$

So, $X(\omega)=\frac{1}{2\pi}\int Y(\theta)\,W(\omega-\theta)\,d\theta$ where

$$\text{DTFT}[y[n]]=Y(\omega)=\pi\,\text{rep}_{2\pi}\left[\delta\left(\omega-\tfrac{5\pi}{10}\right)+\delta\left(\omega+\tfrac{5\pi}{10}\right)\right]$$

$$\text{DTFT}[w[n]]=W(\omega)=\text{psinc}_{10}(\omega)\,e^{-j\omega\frac{10-1}{2}}$$

By the convolution,

$$X(\omega)=\frac{1}{2\pi}\cdot\pi\left[\text{psinc}_{10}\left(\omega-\tfrac{5\pi}{10}\right)e^{-j\left(\omega-\frac{5\pi}{10}\right)\frac{9}{2}}+\text{psinc}_{10}\left(\omega+\tfrac{5\pi}{10}\right)e^{-j\left(\omega+\frac{5\pi}{10}\right)\frac{9}{2}}\right]$$

$, -\pi<\omega<\pi$ with period $=2\pi$

Therefore,

$$X(k)=X(\omega)\big|_{\omega=\frac{2\pi k}{10}},\ k=0,1,\cdots,9$$

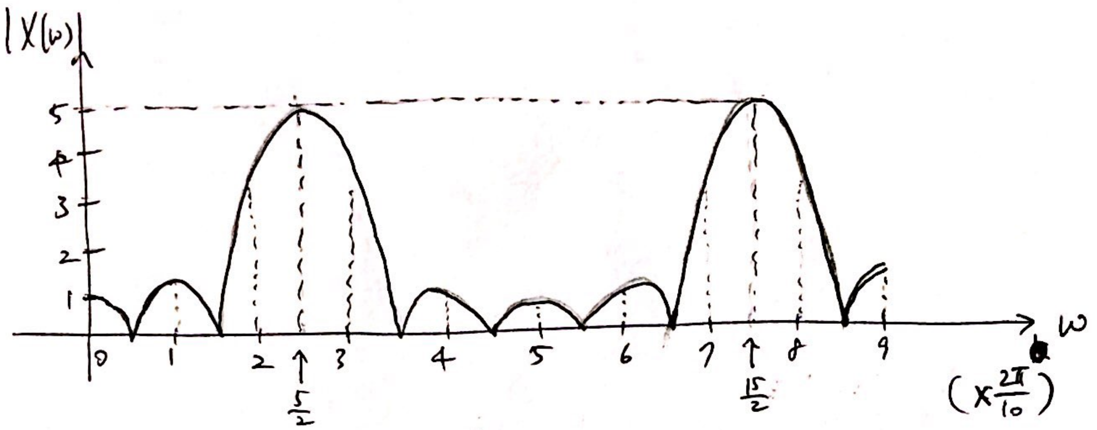
$$=\frac{1}{2}\left[\text{psinc}_{10}\left(\tfrac{2\pi}{10}\left(k-\tfrac{5}{2}\right)\right)e^{-j\frac{2\pi}{10}\left(k-\frac{5}{2}\right)\frac{9}{2}}\right.$$
$$\left.+\,\text{psinc}_{10}\left(\tfrac{2\pi}{10}\left(k+\tfrac{5}{2}\right)\right)e^{-j\frac{2\pi}{10}\left(k+\frac{5}{2}\right)\frac{9}{2}}\right]$$

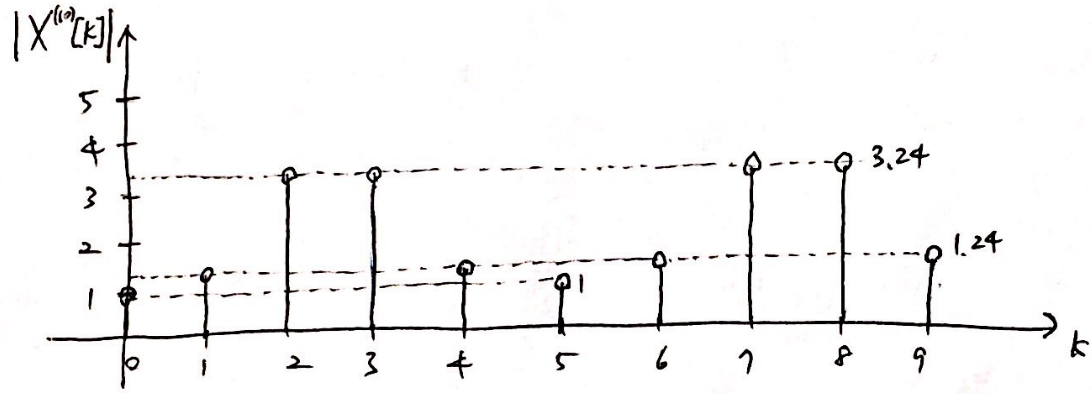
(Particularly, $X(k)=5$ (peak value) when $k=\pm\frac{5}{2},\pm\frac{15}{2},\pm\frac{25}{2},\cdots$.

This can help doing part (b).)

4. (continued)

b)





* The graph should be symmetric about $k=5$.
* Should fully label the axes.
* Plotted at $k = 0, 1, 2, \cdots, 9$
* Either show the peak value of $|X(\omega)|$ or include at least one correct DFT value on $|X^{(10)}[k]|$.
* Highest value at $k = 2, 3, 7, 8$.