17 February 2016 **Name:** Solution

ECE 438 **Exam No. 1** **Spring 2016**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = \frac{1}{4}x[n+1] + \frac{1}{2}x[n] + \frac{1}{4}x[n-1].$$

a. (9) Find the response of this system to the input

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & \text{else} \end{cases}.$$

b. (9) Find a simple expression for the frequency response $H(\omega)$ of this system.

c. (7) Based on your answer to part (a), find simple expressions for the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

a.

n	$y[n]$	$\frac{1}{4}x[n+1]$	$\frac{1}{2}x[n]$	$\frac{1}{4}x[n-1]$
-4	0	0	0	0
-3	1/4	1/4	0	0
-2	3/4	1/4	1/2	1/4
-1	1	1/4	1/2	1/4
0	1	1/4	1/2	1/4
1	1	1/4	1/2	1/4
2	3/4	0	1/2	1/4
3	1/4	0	0	1/4
4	0	0	0	0

Therefore, $y[n] = \begin{cases} 1/4, & \cancel{\text{when}}\ n = -3, 3 \\ 3/4, & n = -2, 2 \\ 1, & n = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$

1. (continued)

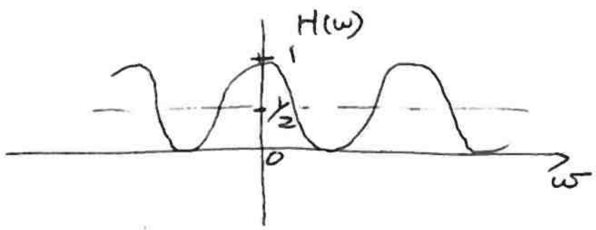
b. Let $x[n] = e^{j\omega n}$, $y[n] = H(\omega) e^{j\omega n}$

$$\Rightarrow y[n] = H(\omega) e^{j\omega n} = \frac{1}{4} e^{j\omega(n+1)} + \frac{1}{2} e^{j\omega n} + \frac{1}{4} e^{-j\omega(n-1)}$$

$$\Rightarrow H(\omega) = \frac{1}{4} e^{j\omega} + \frac{1}{2} + \frac{1}{4} e^{-j\omega}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos(\omega)$$

c.



solution 1: plot $H(\omega)$ and find it's always positive.
Because $H(\omega)$ is positive everywhere,

$$|H(\omega)| = \frac{1}{2} + \frac{1}{2}\cos(\omega)$$

$$\angle H(\omega) = 0$$

solution 2:

$$H(\omega) = \frac{1}{2} + \frac{1}{2}\cos(\omega) = \cos^2\left(\frac{\omega}{2}\right) \text{ ※}$$

$H(\omega)$ is always positive, therefore.

$$|H(\omega)| = \cos^2\left(\frac{\omega}{2}\right) \text{ ※}$$

$$\angle H(\omega) = 0 \text{ ※}$$

2. (25 pts.) Perform the convolution $w[n]$ of the following two signals, and carefully sketch the output signal $w[n]$.

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & 0 \le |n| \le 9 \\ 0, & \text{else} \end{cases} \text{, and } y[n] = u[n].$$

$$w[n] = \sum_k x[k]\, y[n-k]$$

1) $n < -9$: no overlap

$w[n] = 0$

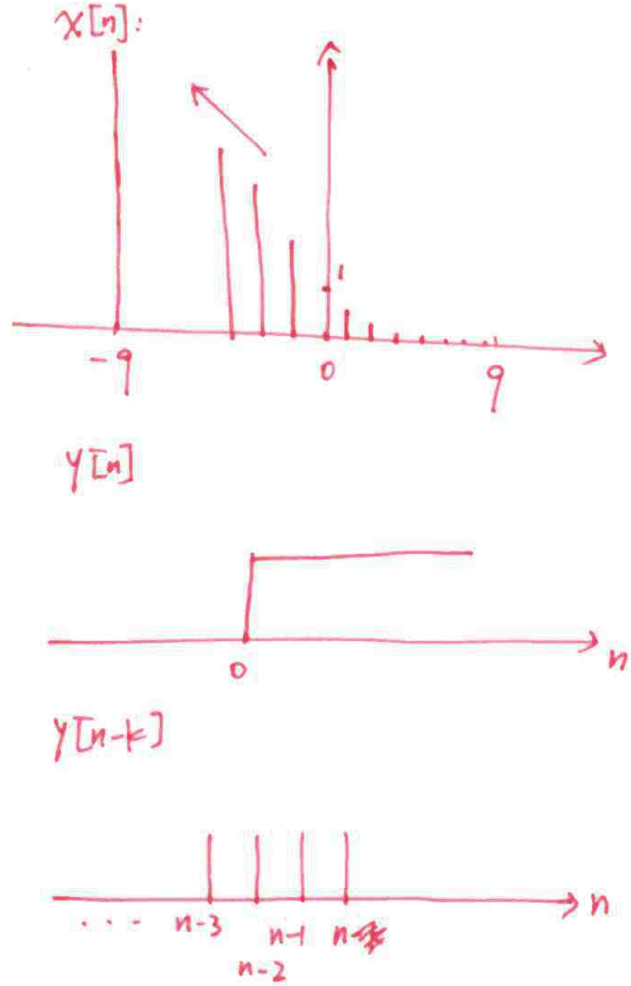
2) $-9 \le n < 9$:

$$w[n] = \sum_{k=-9}^{n} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right)^{-9} \cdot \frac{1-\left(\frac{1}{2}\right)^{n+10}}{\frac{1}{2}}$$

$$= \left(\tfrac{1}{2}\right)^{-10} - \left(\tfrac{1}{2}\right)^{n} = 2^{10} - 2^{-n}$$

3) $n > 9$:

$$w[n] = \left(\tfrac{1}{2}\right)^{-10} - \left(\tfrac{1}{2}\right)^{-9}$$



Plot of output signal is on next page

Plot of output signal is on next page

2. $x[n] = \left(\frac{1}{2}\right)^n, \ 0 \le |n| \le 9$; $y[n] = u[n]$.

$w[n] = \sum_k x[k]\, y[n-k]$.

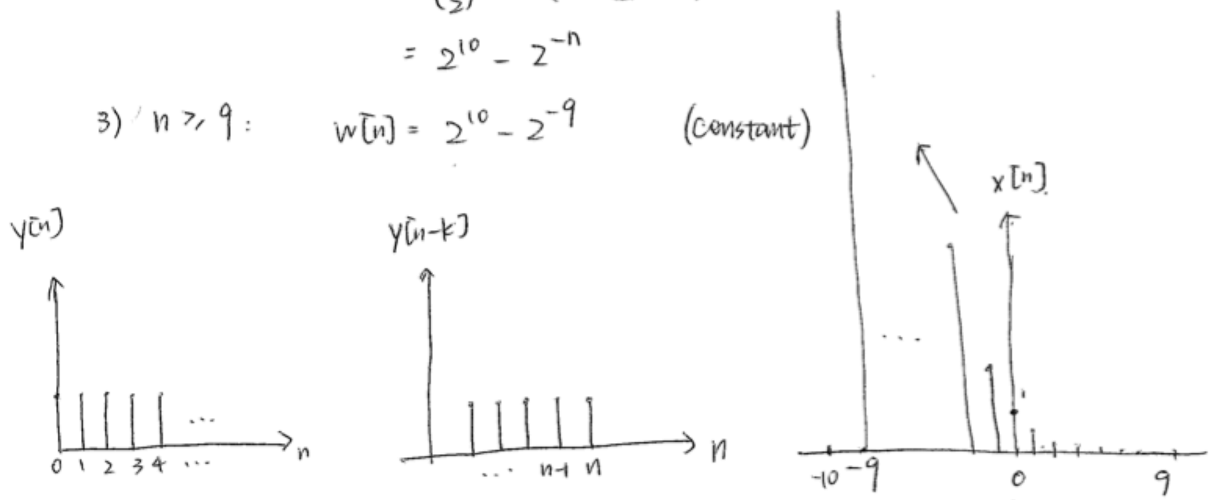
1) $n < -9$: no overlap.

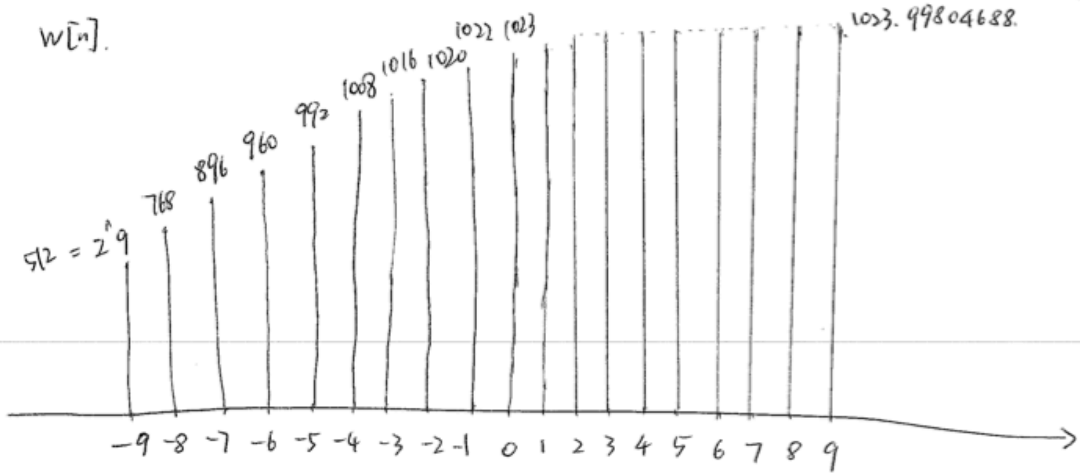
2) $-9 \le n < 9$: $w[n] = \sum_{k=-9}^{n} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{-9} \cdot \frac{1-\left(\frac{1}{2}\right)^{n+10}}{\frac{1}{2}}$

$= \left(\frac{1}{2}\right)^{-10} \cdot \left(1 - \left(\frac{1}{2}\right)^{n+10}\right)$

$= 2^{10} - 2^{-n}$

3) $n \geq 9$: $w[n] = 2^{10} - 2^{-9}$ (constant)

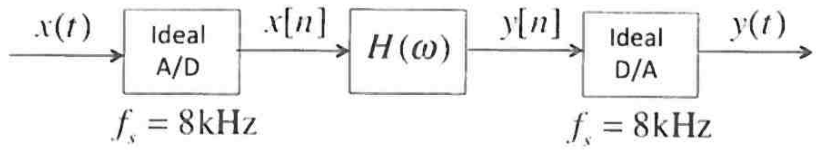




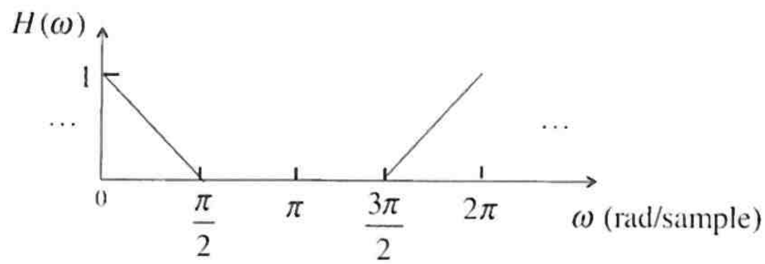
Not in Scale.

2. (continued)

3. (25) Consider the system shown below, which operates at an 8 kHz sampling rate.



Here the digital filter in the center of the system has frequency response $H(\omega)$ given below.



Suppose that the input to this system is

$$x(t) = \cos\left(2\pi(3000)t\right) + \cos\left(2\pi(7000)t\right).$$

Note that the Ideal A/D can be viewed as an ideal low pass filter with cutoff $f_c = 4\text{kHz}$, followed by a comb operation sampling at an $1/8000$ sec. interval. Similarly, the Ideal D/A can be viewed as a train of sample-modulated impulses separated by an $1/8000$ sec. interval (comb operation), followed by an ideal low pass filter with cutoff $f_c = 4\text{kHz}$.

a. (8) Find a simple expression for the DTFT $X(\omega)$ of the signal $x[n]$.

b. (9) Find a simple expression for the DTFT $Y(\omega)$ of the signal $y[n]$.

c. (8) Find a simple expression for the output $y(t)$ of the overall system.

a) after the low-pass filter

$$x_f(t) = \cos(2\pi\, 3000 t)$$

$$X_f(f) = \frac{1}{2}\left(\delta(f-3000) + \delta(f+3000)\right)$$

$$X_d(\omega) = f_s \, \text{rep}_{f_s}\left[X_f(f)\right]\Big|_{f=\frac{\omega f_s}{2\pi}} = f_s \, \text{rep}_{f_s}\Big[\frac{1}{2}\delta\left(\frac{\omega f_s}{2\pi} - 3000\right) + \frac{1}{2}\delta\left(\frac{\omega f_s}{2\pi} + 3000\right)\Big]$$

3. (continued - 1)

$$= \frac{\cancel{f_s}\, 2\pi}{\cancel{f_s}} \cdot \frac{1}{2}\, \text{rep}_{2\pi}\left[\delta\left(\omega - \frac{3\pi}{4}\right) + \delta\left(\omega + \frac{3\pi}{4}\right)\right]$$

$$= \pi\, \text{rep}_{2\pi}\left[\delta\left(\omega - \frac{3\pi}{4}\right) + \delta\left(\omega + \frac{3\pi}{4}\right)\right]$$

b) $H\left(\frac{3\pi}{4}\right) = 0$

$Y(\omega) = 0$

c) $y(t) = 0$

Alternative for a)

$$x[n] = x_f(t)\Big|_{t = \frac{1}{8000}n}$$

$$= \cos\left(\frac{2\pi}{8000} 3000 n\right)$$

$$= \cos\left(\frac{3\pi}{4} n\right)$$

Then using $\cos(\omega_0 n) \xleftrightarrow{DTFT} \pi\, \text{rep}_{2\pi}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$

$$x[n] \xleftrightarrow{DTFT} \pi\, \text{rep}_{2\pi}\left[\delta\left(\omega - \frac{3\pi}{4}\right) + \delta\left(\omega + \frac{3\pi}{4}\right)\right] = X(\omega)$$

3. (continued - 2)

4. (25 pts) Consider the signal:

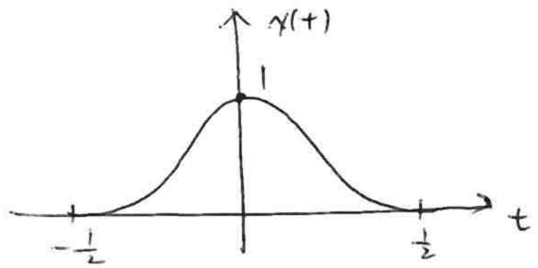
$$x(t) = \begin{cases} \frac{1}{2}\left(1+\cos(2\pi t)\right), & |t| < \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

a) (5) Carefully sketch $x(t)$ being careful to dimension all important quantities.

b) (12) Find a simple closed-form expression for the CTFT $X(f)$ of $x(t)$. Your answer should not contain any operators.

c) (8) Carefully sketch $X(f)$. Be sure to dimension all important quantities.

(a)



(b) $x(t) = \frac{1}{2}(1+\cos(2\pi t))\,\mathrm{rect}(t)$

$X(f) = \left[\frac{1}{2}\delta(f) + \frac{1}{4}\delta(f-1) + \frac{1}{4}\delta(f+1)\right] * \mathrm{sinc}(f)$

$= \frac{1}{2}\mathrm{sinc}(f) + \frac{1}{4}\mathrm{sinc}(f-1) + \frac{1}{4}\mathrm{sinc}(f+1)$

(c) Thinking that $\mathrm{sinc}(f)$ is 1 at $f=0$ and 0 at $f=$ the other integers,

When $f=0$ $X(f) = \frac{1}{2} + 0 + 0 = \frac{1}{2}$

$f=1$ $X(f) = 0 + \frac{1}{4} + 0 = \frac{1}{4}$

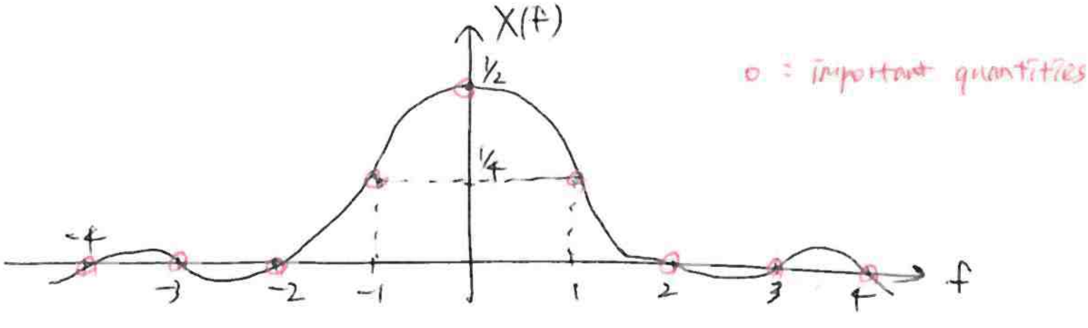
$f=2$ $X(f) = 0 + 0 + 0 = 0$

$f=3$ $X(f) = 0 + 0 + 0 = 0$

$\vdots$

4. (continued)

Also we know the function is even.



* comment : Alternatively, you can draw each sinc function and sum them up.