6 May 2014 **Name:** SOLUTION **Signature:** ________________

ECE 438 **Final Exam** **Spring 2014**

- You have 120 minutes to work the following five problems.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem. Please note that the problems are arranged in the order that the topics were covered during the semester, not necessarily in the order of difficulty to solve them.

1. (25 pts.) Consider a system described by the following equation

$$y[n] = x[n] - x[n-2].$$

a. (6) Find the response $y[n]$ to the following input

$$x[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{else} \end{cases}.$$

b. (8) Find a simple expression for the frequency response $H(\omega)$ for this system.

c. (6) From your answer to part (b), determine simple expressions for the magnitude and phase of the frequency response.

d. (5) Using the frequency response $H(\omega)$ that you obtained in part (b) above, find as simple an expression as possible for the response of the system to the input

$$x[n] = \cos(\pi n / 4).$$

Do NOT use the system equation provided at the beginning of this problem to answer part (d).

Solution for problem #1

a. $y[0] = 1$

$y[1] = 1$

$y[6] = -1$

$y[7] = -1$

$y[n] = 0$, otherwise

$\therefore y[n] = \delta[n] + \delta[n-1] - \delta[n-6] - \delta[n-7]$

b. $h[n] = \delta[n] - \delta[n-2]$

$$H(w) = 1 - e^{-j2w} = e^{-jw}(e^{jw} - e^{-jw}) = 2e^{-jw} \cdot \frac{e^{jw} - e^{-jw}}{2}$$

$$= 2j e^{-jw} \sin w$$

$$\therefore H(w) = 1 - e^{-j2w} = 2j e^{-jw} \sin w$$

c. magnitude of frequency response :

$$|H(w)| = |2j e^{-jw} \sin w| = |2 \sin w|$$

$\therefore |2 \sin w|$

phase of frequency response :

$$\measuredangle H(w) = \measuredangle 2 + \measuredangle j + \measuredangle e^{-jw} + \measuredangle \sin w$$

$$= 0 + \pi/2 + (-w) + \measuredangle \sin w$$

$$\measuredangle \sin w = \begin{cases} 0, & 0 \le w < \pi \\ \pi, & \pi \le w < 2\pi \end{cases}$$

$$\therefore \measuredangle H(w) = \frac{\pi}{2} - w, \quad 0 \le w < \pi$$

$$\frac{3\pi}{2} - w, \quad \pi \le w < 2\pi$$

or

$$-\frac{\pi}{2} - w, \quad -\pi \le w < 0$$

d.

$$\cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n}$$

For an LTI system

$$e^{j\omega_0 n} \rightarrow \boxed{H(\omega)} \rightarrow H(\omega_0)e^{j\omega_0 n}$$

$\omega_0 = \pi/4$ and $-\pi/4$ for two terms of the input

$$y[n] = \frac{1}{2}H\left(\frac{\pi}{4}\right)e^{j\frac{\pi}{4}n} + \frac{1}{2}H\left(-\frac{\pi}{4}\right)e^{-j\frac{\pi}{4}n}$$

$$= \left(1-e^{-j2\left(\frac{\pi}{4}\right)}\right)\frac{e^{j\frac{\pi}{4}n}}{2} + \left(1-e^{j2\left(\frac{\pi}{4}\right)}\right)\frac{e^{-j\frac{\pi}{4}n}}{2}$$

$$= -\sqrt{2}\sin\left(\frac{\pi}{4}(n-1)\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}(n+1)\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}(n-7)\right)$$

2. (25 pts.) Fast Fourier Transform (FFT)

a. (1) Write down the equation for a 12-point DFT.

b. (2) Based on your answer to part (a), determine the approximate number of complex operations needed to directly evaluate a 12-point DFT. Here a complex operation consists of one complex multiplication plus one complex addition.

c. (9) Derive a complete set of equations for the 12-point FFT.

d. (8) Based on your answer to part (c), draw a complete flow diagram for a 12-point FFT.

e. (5) From your flow diagram in part (d), determine the approximate number of complex operations required to compute a 12-point FFT.

a) $X^{12}[k] = \sum_{n=0}^{11} x[n] \, e^{-\frac{j2\pi kn}{12}}$

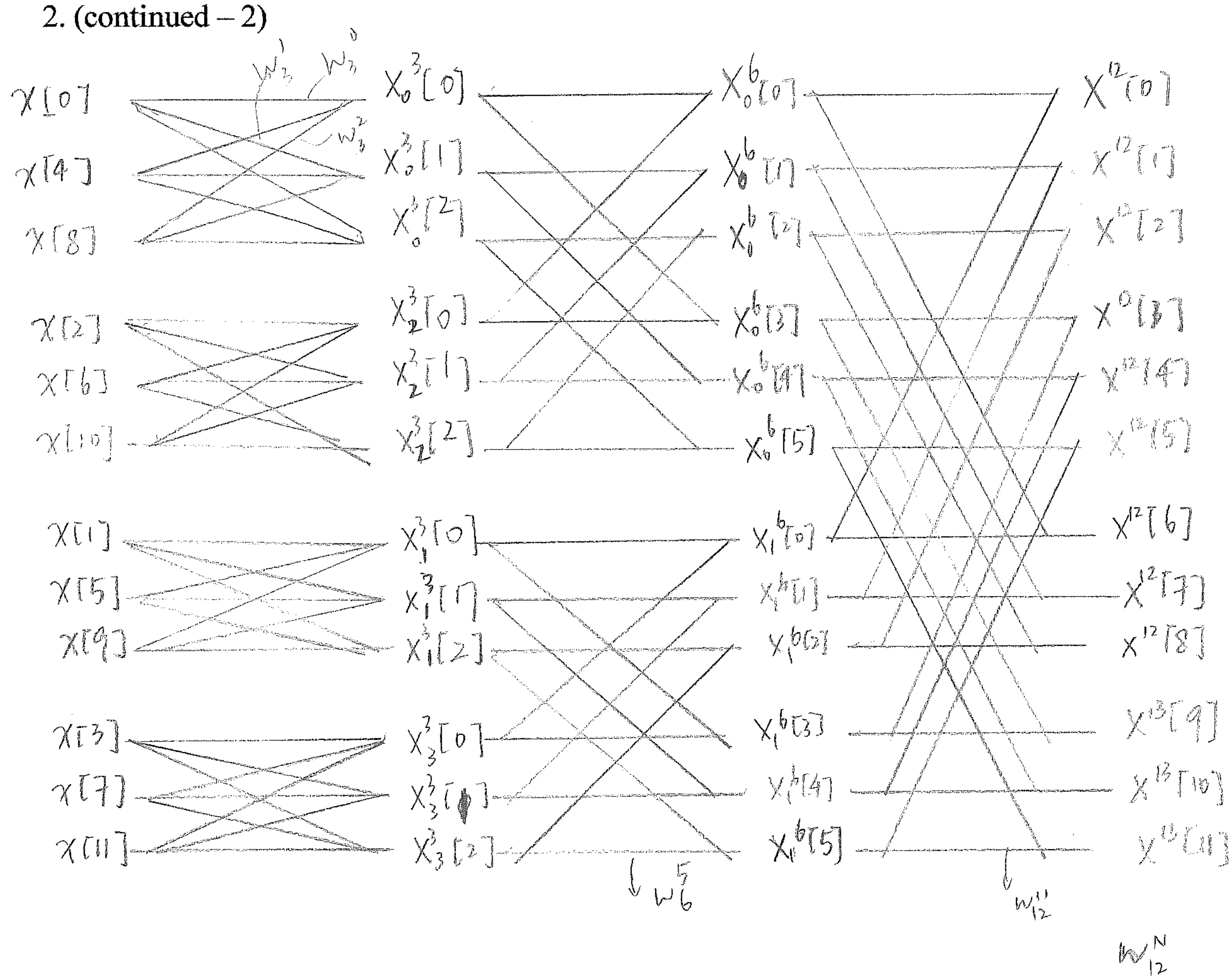
b) $12^2 = 144$.

c). $X^{12}[k] = \sum_{p=1}^{1} \sum_{\ell=1}^{1} \sum_{m=0}^{2} x[4m+2\ell+p] \, e^{-j\frac{2\pi k(4m+2\ell+p)}{12}}$

$\Rightarrow X^{12}[k] = \sum_{p=0}^{1} \sum_{\ell=0}^{1} \sum_{m=0}^{2} x[4m+2\ell+p] \, e^{-j\frac{2\pi k 4m}{12}} \cdot e^{-j\frac{2\pi k \cdot 2\ell}{2}} \cdot e^{-j\frac{2\pi kp}{12}}$

$$X^{12}[k] = \underbrace{\sum_{p=0}^{1} e^{\frac{j2\pi kp}{12}} \underbrace{\sum_{\ell=0}^{1} e^{-j\frac{2\pi k\ell}{6}} \underbrace{\sum_{m=0}^{2} x[4m+2\ell+p] \, e^{-j\frac{2\pi km}{3}}}_{3\text{ pt DFT}}}_{6\text{pt DFT}}}_{12\text{pt DFT}}$$

2. (continued – 2)



e) $12 + 12 + 2 \times 12 = 48$ COs.

3. (25) Consider a wide-sense stationary random signal $X[n]$ with mean 0 and autocorrelation

$$r_{XX}[n] = \begin{cases} 1, & n = 0, \\ \frac{1}{2}, & |n| = 1, \\ 0, & \text{else}. \end{cases}$$

Suppose we filter this signal with two different filters to generate two new random signals $Y[n]$ and $Z[n]$, according to

$$Y[n] = \frac{1}{2}\left(X[n] + X[n-1]\right),$$
$$Z[n] = X[n] - X[n-1].$$

a. (1) Find the mean $\mu_Y[n] = E\left(Y[n]\right)$ of the first output signal.

b. (12) Find the autocorrelation $r_{ZZ}[n] = E\left(Z[m]Z[m+n]\right)$ of the second output signal.

c. (12) Find the cross-correlation $r_{YZ}[n] = E\left(Y[m]Z[m+n]\right)$ between the first and second output signals.

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(a) $\mu_Y[n] = E(Y[n]) = E\left(\frac{1}{2}(X[n] + X[n-1])\right) = \frac{1}{2}E(X[n]) + \frac{1}{2}E(X[n-1])$

Since X is wide-sense stationary $\Rightarrow E(X[n-1]) = E(X[n]) = 0$

$\Rightarrow \mu_Y[n] = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0$

(b) $z[n] = x[n] - x[n-1] = h[n] * x[n] \Rightarrow h[n] = \delta[n] - \delta[n-1] = \{\underset{\uparrow_0}{1}, -1\}$

$r_{zz}[n] = h[n] * r_{xz}[-n] = h[n] * (h[-n] * r_{xx}[-n]) = h[n] * h[-n] * r_{xx}[-n]$

$h[-n] = \{-1, \underset{\uparrow_0}{1}\}$, $r_{xx}[-n] = \{\frac{1}{2}, \underset{\uparrow_0}{1}, \frac{1}{2}\}$

$\Rightarrow r_{zz}[n] = \{\underset{\uparrow_0}{1}, -1\} * \{-1, \underset{\uparrow_0}{1}\} * \{\frac{1}{2}, \underset{\uparrow_0}{1}, \frac{1}{2}\} = \{-1, \underset{\uparrow_0}{2}, -1\} * \{\frac{1}{2}, \underset{\uparrow_0}{1}, \frac{1}{2}\}$

$= \{-\frac{1}{2}, 0, \underset{\uparrow_0}{1}, 0, -\frac{1}{2}\}$

(c) $r_{YZ}[n] = E(Y[m]Z[m+n]) = E\left(\frac{1}{2}(X[m] + X[m-1])(X[m+n] - X[m+n-1])\right)$

$= \frac{1}{2}E(X[m]X[m+n]) - \frac{1}{2}E(X[m]X[m+n-1]) + \frac{1}{2}E(X[m-1]X[m+n])$

$- \frac{1}{2}E(X[m-1]X[m+n-1])$

$= \frac{1}{2}r_{xx}[n] - \frac{1}{2}r_{xx}[n-1] + \frac{1}{2}r_{xx}[n+1] - \frac{1}{2}r_{xx}[n]$

$= \frac{1}{2}r_{xx}[n+1] - \frac{1}{2}r_{xx}[n-1]$

$= \frac{1}{2}\{\frac{1}{2}, 1, \underset{\uparrow_0}{\frac{1}{2}}\} - \frac{1}{2}\{\underset{\uparrow_0}{\frac{1}{2}}, 1, \frac{1}{2}\} = \{\frac{1}{4}, \frac{1}{2}, \underset{\uparrow_0}{0}, -\frac{1}{2}, -\frac{1}{4}\}$

4. (25 pts) Consider the signal

$$x(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}.$$

We wish to approximate $x(t)$ by the signal

$$\hat{x}(t) = a_0 x_0(t) + a_1 x_1(t),$$

where a_0 and a_1 are constants, and $x_0(t)$ and $x_1(t)$ are basis functions given by

$$x_0(t) = 1, \quad x_1(t) = t.$$

a. (15) Find optimal values for a_0 and a_1 that will minimize the total squared error

$$\Phi = \int_0^1 |\hat{x}(t) - x(t)|^2 dt.$$

b. (5) Carefully plot $\hat{x}(t)$ for your optimal coefficients a_0 and a_1, and the original signal $x(t)$ on the same axes. Be sure to dimension your axes.

c. (5) Compute the minimum RMS approximation error, which in this case is $\sqrt{\Phi}$.

a. $\Phi = \int_0^1 |\hat{x}(t) - x(t)|^2 dt = \int_0^1 (a_0 + a_1 t - t^2)^2 dt$

$$\frac{\partial \Phi}{\partial a_0} = 2\int_0^1 (a_0 + a_1 t - t^2)\, dt = 2\left[a_0 t + \frac{1}{2} a_1 t^2 - \frac{1}{3} t^3 \right]_0^1$$

$$= 2\left(a_0 + \frac{1}{2} a_1 - \frac{1}{3}\right) = 0 \qquad ①$$

$$\frac{\partial \Phi}{\partial a_1} = 2\int_0^1 (a_0 + a_1 t - t^2) \cdot t\, dt = 2\int_0^1 (a_0 t + a_1 t^2 - t^3)\, dt$$

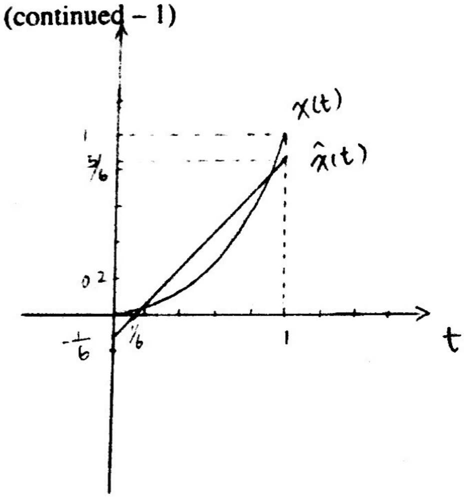
$$= 2\left[\frac{1}{2} a_0 t^2 + \frac{1}{3} a_1 t^3 - \frac{1}{4} t^4 \right]_0^1$$

$$= 2\left(\frac{1}{2} a_0 + \frac{1}{3} a_1 - \frac{1}{4}\right) = 0 \qquad ②$$

Solve ① ②, get $\boxed{\begin{array}{l} a_0 = -\frac{1}{6} \\ a_1 = 1 \end{array}}$

4. (continued – 1)

b



$\hat{x}(t) = t - \frac{1}{6}$

$t = 0 \quad \hat{x}(t) = -\frac{1}{6}$

$t = 1 \quad \hat{x}(t) = \frac{5}{6}$

c

$$\Phi = \int_0^1 \left(t - \frac{1}{6} - t^2 \right)^2 dt$$

$$= \int_0^1 \left(t^2 + \frac{1}{36} + t^4 - \frac{1}{3}t - 2t^3 + \frac{1}{3}t^2 \right) dt$$

$$= \int_0^1 \left(\frac{4}{3}t^2 + \frac{1}{36} + t^4 - \frac{1}{3}t - 2t^3 \right) dt$$

$$= \left[\frac{4}{9}t^3 + \frac{1}{36}t + \frac{1}{5}t^5 - \frac{1}{6}t^2 - \frac{1}{2}t^4 \right]_0^1$$

$$= \frac{4}{9} + \frac{1}{36} + \frac{1}{5} - \frac{1}{6} - \frac{1}{2} = \frac{1}{180}$$

$$\sqrt{\Phi} = \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}} = \frac{\sqrt{5}}{30}$$

5. (25 pts) Consider the 2D signal $f(x,y)$

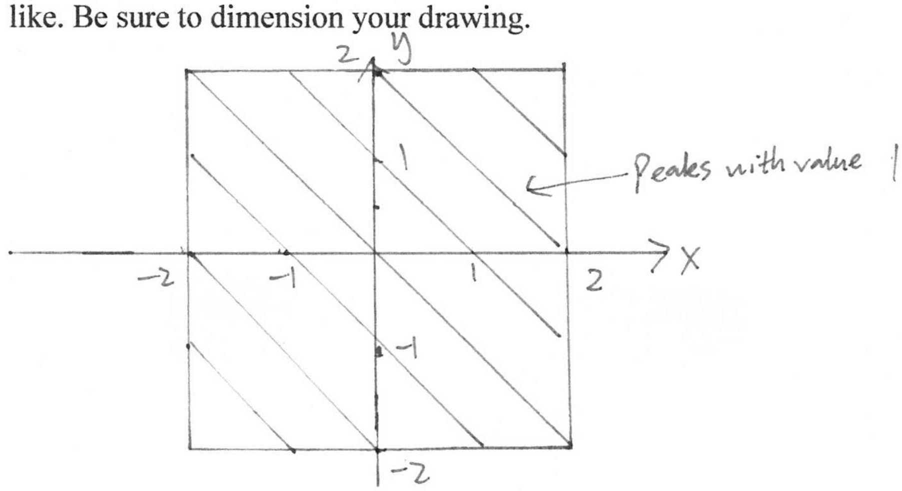
$$f(x,y)=\begin{cases}\cos\big(2\pi(x+y)\big), & |x|,|y|\le 2,\\ 0, & \text{else.}\end{cases}$$

a. (7) Sketch $f(x,y)$ with sufficient detail to show that you know what it looks like. Be sure to dimension your drawing.

b. (11) Find a simple expression for its CSFT $F(u,v)$ using transform pairs and properties. Your answer should not contain any operators other than summations.

c. (7) Sketch $F(u,v)$ with sufficient detail to show that you know what it looks like. Be sure to dimension your drawing.

a.



b. $f(x,y) = \cos(2\pi(x+y)) \cdot \mathrm{rect}\left(\frac{x}{4}, \frac{y}{4}\right)$

$= f_1(x,y) \cdot f_2(x,y)$

$F_1(u,v) = \frac{1}{2}[\delta(u-1, v-1) + \delta(u+1, v+1)]$

$F_2(u,v) = 16\,\mathrm{sinc}(4u, 4v)$

$F(u,v) = F_1(u,v) ** F_2(u,v)$

$= 8\left[\mathrm{sinc}(4(u-1), 4(v-1)) + \mathrm{sinc}(4(u+1), 4(v+1))\right]$

c.