23 April 2014 **Name:** Solution

ECE 438 **Exam No. 3** **Spring 2014**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider a random variable X with density function

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

a. (8) Find the mean and variance of X.

Suppose we generate a new random variable $Y = Q(X)$ by quantizing X according to the following quantizer:

$$Q(x) = \begin{cases} y_0, & 0 \le x \le 0.5 \\ y_1, & 0.5 \le x \le 1 \end{cases}$$

b. (8) Determine the values for y_0 and y_1 that will minimize the mean-squared quantization error $\phi = E\left\{|Y - X|^2\right\}$ for the given threshold $x_0 = 0.5$. Note that you do not need to prove that these values for y_0 and y_1 will minimize ϕ. You just need to calculate what they are.

c. (7) Assume that $y_0 = 0.25$ and $y_1 = 0.75$. Determine the correlation coefficient ρ_{XY} between X and Y.

1 (a) $E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2}{3}$

$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2 \cdot 2x\,dx = \frac{1}{2}$

$\sigma_X^2 = E\{X^2\} - (E\{X\})^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

(b) $$y_0 = \frac{\int_0^{0.5} x f_X(x)\,dx}{\int_0^{0.5} f_X(x)\,dx} = \frac{\int_0^{0.5} 2x^2\,dx}{\int_0^{0.5} 2x\,dx} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

$$y_1 = \frac{\int_{0.5}^1 x f_X(x)\,dx}{\int_{0.5}^1 f_X(x)\,dx} = \frac{\int_{0.5}^1 2x^2\,dx}{\int_{0.5}^1 2x\,dx} = \frac{\frac{7}{12}}{\frac{3}{4}} = \frac{7}{9}$$

(c) $E\{XY\} = E\{X\,Q(X)\} = \int_{-\infty}^{\infty} x\,Q(x)\,f_X(x)\,dx$

$\int_0^{0.5} 0.25x \cdot 2x\,dx + \int_{0.5}^1 0.75x \cdot 2x\,dx$

$\frac{1}{6}x^3\Big|_0^{0.5} + \frac{1}{2}x^3\Big|_{0.5}^1 = \frac{11}{24}$

$E\{Y\} = E\{Q(X)\} = \int_{-\infty}^{\infty} Q(x)\,f_X(x)\,dx$

$= \int_0^{0.5} 0.25 \cdot 2x\,dx + \int_{0.5}^1 0.75 \cdot 2x\,dx$

$= \frac{1}{4}x^2\Big|_0^{0.5} + \frac{3}{4}x^2\Big|_{0.5}^1 = \frac{5}{8}$

$E\{Y^2\} = \int_{-\infty}^{\infty} Q^2(x)\,f_X(x)\,dx = \int_0^{0.5} 0.25^2 \cdot 2x\,dx + \int_{0.5}^1 0.75^2 \cdot 2x\,dx = \frac{7}{16}$

$\sigma_Y^2 = E\{Y^2\} - E\{Y\}^2 = \frac{7}{16} - \frac{25}{64} = \frac{3}{64}$

$$\rho_{XY} = \frac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_X \sigma_Y} = \frac{\frac{11}{24} - \frac{2}{3}\cdot\frac{5}{8}}{\sqrt{\frac{1}{18}\cdot\frac{3}{64}}} = \frac{\frac{1}{24}}{\frac{1}{8}\sqrt{\frac{1}{6}}} = \frac{\sqrt{6}}{3}$$

2. (25) Let $X[n]$ be a wide-sense stationary sequence of random variables with zero mean and autocorrelation function

$$r_{XX}[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & |n| = 1 \\ 0, & \text{else} \end{cases}$$

Suppose that this sequence is filtered to generate the output sequence

$$y[n] = x[n] - x[n-1]$$

a. (3) Find the mean of the sequence $Y[n]$.

b. (12) Find the cross-correlation $r_{XY}[n]$ between X and Y.

c. (10) Find the autocorrelation $r_{YY}[n]$ of the output Y.

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(a) $E\{Y(n)\} = E\{X[n] - X[n-1]\} = E\{X[n]\} - E\{X[n-1]\}$

Since $X[n]$ is wide-sense stationary

$\Rightarrow E\{X[n]\} = E\{X[n-1]\} = 0$

$\Rightarrow E\{Y(n)\} = 0 - 0 = 0$ *

(b) $Y[n] = X[n] - X[n-1] = h[n] * X[n] \Rightarrow h[n] = \delta[n] - \delta[n-1]$

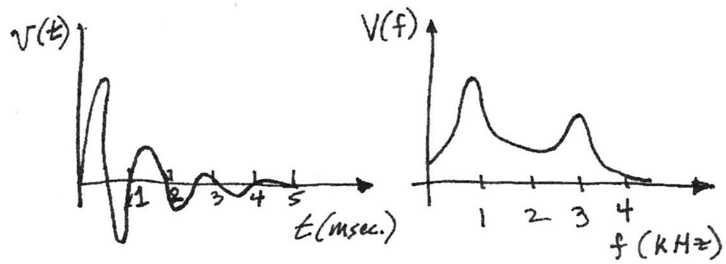
$r_{XY}[n] = h[n] * r_{XX}[n] = \sum_{m=-\infty}^{\infty} h[m]\, r_{XX}[n-m] = r_{XX}[n] - r_{XX}[n-1]$

$= \{\tfrac{1}{2}, \underset{\uparrow 0}{1}, \tfrac{1}{2}\} - \{\underset{\uparrow 0}{\tfrac{1}{2}}, 1, \tfrac{1}{2}\} = \{\tfrac{1}{2}, \underset{\uparrow 0}{\tfrac{1}{2}}, -\tfrac{1}{2}, -\tfrac{1}{2}\}$ *

(c) $r_{YY}[n] = h[n] * r_{XY}[-n] = \sum_{m=-\infty}^{\infty} h[m]\, r_{XY}[m-n] = r_{XY}[-n] - r_{XY}[1-n]$

$= \{-\tfrac{1}{2}, -\tfrac{1}{2}, \underset{\uparrow 0}{\tfrac{1}{2}}, \tfrac{1}{2}\} - \{-\tfrac{1}{2}, \underset{\uparrow 0}{-\tfrac{1}{2}}, \tfrac{1}{2}, \tfrac{1}{2}\} = \{-\tfrac{1}{2}, 0, \underset{\uparrow 0}{1}, 0, -\tfrac{1}{2}\}$ *

3. (25) Consider a voiced phoneme for which the time-domain continuous-time vocal tract response $v(t)$ and corresponding frequency response (CTFT) $V(f)$ are given below.



a. (7) Assume that the pitch frequency for the speaker is 100 Hz. Sketch what the continuous-time domain speech waveform $s(t)$ would look like in this case. Be sure to dimension all important quantities in the speech waveform.

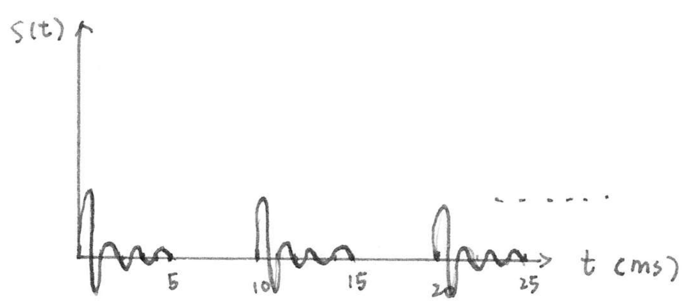
b. (9) Suppose that we sample the speech waveform $s(t)$ above at an 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 25 samples. Carefully sketch the resulting spectrogram as a function of discrete-time index n and digital frequency ω (radians/sample). Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

10ms · 10K

$\frac{1 \times 2\pi}{10K}$.

c. (9) Suppose that we sample the speech waveform $s(t)$ above at a 10 kHz rate, and compute the short-time discrete-time Fourier transform (STDTFT) using a window of length 1,000 samples. Carefully sketch the resulting spectrogram as a function of discrete-time index n and digital frequency ω (radians/sample). Be sure to dimension all important quantities in your plot. Is this a wide-band or narrow-band spectrogram?

a. Pitch Period $= \frac{1}{100\text{Hz}} = 10$ ms. $s(t) = v(t) * e(t)$.

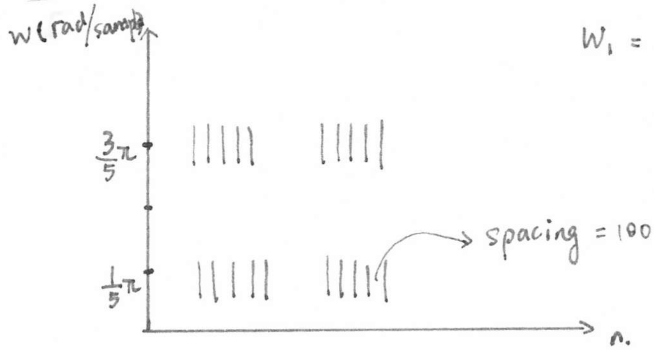


3. (continued)

(b) According to the given $V(f)$.

The formant frequencies are $F_1 = 1\text{ KHz}$ and $F_2 = 3\text{ KHz}$.

$W_1 = 1 \cdot \frac{2\pi}{10} = \frac{\pi}{5}$ and $W_2 = 3 \cdot \frac{2\pi}{10} = \frac{3}{5}\pi$.



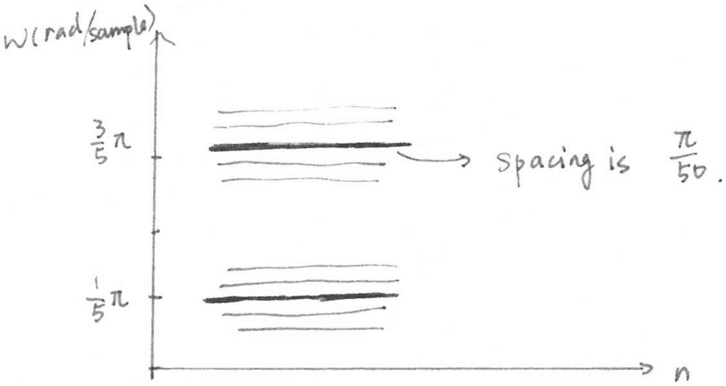
Pitch Period in sample is

$10\text{ms} \cdot 10\text{kHz} = 100$.

This is a wideband spectrogram. The shorter ($25 < 100$) window length gives good time domain resolution but poor frequency domain resolution.

(c)

$W_1 = \frac{\pi}{5}$, $W_2 = \frac{3}{5}\pi$.



Pitch digital frequency:

$100\text{ Hz} \cdot \frac{2\pi}{10\text{KHz}} = \frac{\pi}{50}$.

This is a narrowband spectrogram. The long (time window length gives good frequency domain resolution but poor time domain resolution.

4. (25) Consider the signal

$$x[n] = \begin{cases} \cos(\pi n/8), & n < 0 \\ \cos(\pi n/3), & n \geq 0 \end{cases},$$

Assume a rectangular window

$$w[n] = \begin{cases} 1, & |n| < 25 \\ 0, & \text{else} \end{cases}.$$

a. Compute the STDTFT as defined below

$$X(\omega, n) = \sum_k x[k]w[n-k]e^{-jwk}$$

for the following cases (Be sure to express your answer in terms of the function $\text{psinc}_N(\omega) @ \frac{\sin(\omega N/2)}{\sin(\omega/2)}$ for appropriate values of N:

(6) i. $n < -25$

(6) ii. $n > 25$

(6) iii. $n = 0$

(7) b. Sketch $|X(\omega,n)|$ for all n. Be sure to label important dimensions.

a. i. $X(w,n) = \sum_{k=n-24}^{n+24} \cos\left(\frac{\pi k}{8}\right) e^{-jwk}$

$$= \sum_{k=-\infty}^{\infty} \cos\left(\frac{\pi k}{8}\right) w(n-k)\, e^{-jwk}$$

$$= \text{DTFT}\left\{\cos\left(\frac{\pi k}{8}\right) \cdot w(n-k)\right\}$$

$$\cos\left(\frac{\pi k}{8}\right) \xrightarrow{\text{DTFT}} \pi\, \text{rep}_{2\pi}\left[\delta\left(w - \frac{\pi}{8}\right) + \delta\left(w + \frac{\pi}{8}\right)\right]$$

$$w(n-k) \xrightarrow{\text{DTFT}} W(-w)e^{-jwn} = \text{psinc}_{49}(-w)\, e^{-jwn}$$

$$= \text{psinc}_{49}(w)\, e^{-jwn}$$

$$X(w,n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi\, \text{rep}_{2\pi}\left[\delta\left(\mu - \frac{\pi}{8}\right) + \delta\left(\mu + \frac{\pi}{8}\right)\right] \cdot \text{psinc}_{49}(w-\mu)\, e^{-j(w-\mu)n}\, d\mu$$

$$= \frac{1}{2}\left[\text{psinc}_{49}\left(w - \frac{\pi}{8}\right) e^{-j(w-\frac{\pi}{8})n} + \text{psinc}_{49}\left(w + \frac{\pi}{8}\right) e^{-j(w+\frac{\pi}{8})n}\right]$$

4. (continued - 2)

ii. $n > 25$

$$X(w,n) = \sum_{k=n-24}^{n+24} \cos\left(\frac{\pi n}{3}\right) e^{-jwk}$$

Similar to part i:

we get $X(w,n) = \frac{1}{2}\left[\text{psinc}_{49}\left(w - \frac{\pi}{3}\right) e^{-j(w-\frac{\pi}{3})n} + \text{psinc}_{49}\left(w + \frac{\pi}{3}\right) e^{-j(w+\frac{\pi}{3})n}\right]$

iii. $n=0$

$$X(w,0) = \sum_{k=-24}^{-1} \cos\left(\frac{\pi k}{8}\right) e^{-jwk} + \sum_{k=0}^{24} \cos\left(\frac{\pi k}{3}\right) e^{-jwk}$$

$$\sum_{k=-24}^{-1} \cos\left(\frac{\pi k}{8}\right) e^{-jwk} = \sum_{k} \cos\left(\frac{\pi n}{8}\right) w_1[k] e^{-jwk},$$

where $w_1(k) = \begin{cases} 1 & -24 \le k \le -1 \\ 0 & \text{else} \end{cases}$

$$W_1(w) = \sum_{k=-24}^{-1} e^{-jwk} = e^{-jw\cdot(-24)} \cdot \frac{1-e^{-j24w}}{1-e^{-jw}} = e^{-j\frac{25}{2}w}\,\text{psinc}_{24}(w)$$

Using the DTFT product relation,

$$\sum_{k=-24}^{-1} \cos\left(\frac{\pi k}{8}\right) w_1(k) e^{-jwk} = \frac{1}{2}\left[e^{-j\frac{25}{2}(w-\frac{\pi}{8})}\,\text{psinc}_{24}\left(w-\frac{\pi}{8}\right) + e^{-j\frac{25}{2}(w+\frac{\pi}{8})}\,\text{psinc}_{24}\left(w+\frac{\pi}{8}\right)\right]$$

Similarly, $\displaystyle\sum_{k=0}^{24} \cos\left(\frac{\pi k}{3}\right) e^{-jwk}$

$$= \sum_{k=0}^{24} \cos\left(\frac{\pi k}{3}\right) w_2(k) e^{-jwk}, \text{ where } w_2(k) = \begin{cases} 1 & 0 \le k \le 24 \\ 0 & \text{else} \end{cases}$$

$$W_2(w) = e^{-j12w}\,\text{psinc}_{25}(w)$$

so $\displaystyle\sum_{k=0}^{24} \cos\left(\frac{\pi k}{3}\right) e^{-jwk} = \frac{1}{2}\left[e^{-j12(w+\frac{\pi}{3})}\,\text{psinc}_{25}\left(w+\frac{\pi}{3}\right) + e^{-j12(w-\frac{\pi}{3})}\,\text{psinc}_{25}\left(w-\frac{\pi}{3}\right)\right]$

$$\Rightarrow X(w,0) = \frac{1}{2}\left[e^{-j\frac{25}{2}(w-\frac{\pi}{8})}\,\text{psinc}_{24}\left(w-\frac{\pi}{8}\right) + e^{-j\frac{25}{2}(w+\frac{\pi}{8})}\,\text{psinc}_{24}\left(w+\frac{\pi}{8}\right)\right]$$
$$+ \frac{1}{2}\left[e^{-j12(w-\frac{\pi}{3})}\,\text{psinc}_{25}\left(w-\frac{\pi}{3}\right) + e^{-j12(w+\frac{\pi}{3})}\,\text{psinc}_{25}\left(w+\frac{\pi}{3}\right)\right]$$

4. (continued - 1)

The fact that these individual "pitch" periods can be seen can be explained by looking at what is happening as the window slides along as a function of the time index n. However, one might at first expect that the spectrogram should just be wideband, since the window is long relative to the period of the signals being analyzed.

b.