26 March 2014 **Name:** Solution

ECE 438 **Exam No. 2** **Spring 2014**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the Z-transform $$X(z) = \frac{1}{\left(1-2z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.$$

   Find all possible regions of convergence for $X(z)$ and determine the corresponding time-domain signal $x[n]$ for each possible region of convergence.

Problem 1 solution:

$$X(z) = \frac{A}{(1-2z^{-1})} + \frac{B}{(1-\frac{1}{2}z^{-1})} \qquad \cdots 1$$

$$= \frac{\frac{4}{3}}{(1-2z^{-1})} + \frac{-\frac{1}{3}}{(1-\frac{1}{2}z^{-1})} \qquad \cdots 3$$

Three possibilities for ROC,

- ROC 1: $|z| > 2$ $\cdots 3$

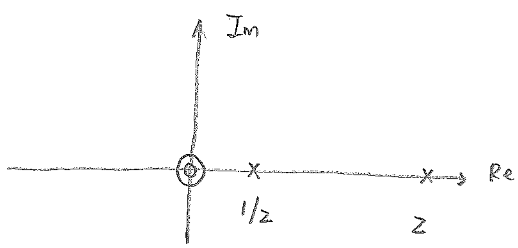
$$x[n] = \frac{4}{3} 2^n u[n] - \frac{1}{3}(\tfrac{1}{2})^n u[n] \qquad \cdots 4$$

- ROC 2: $1/2 < |z| < 2$ $\cdots 3$

$$x[n] = \frac{4}{3}(-2^n u[-n-1]) - \frac{1}{3}(\tfrac{1}{2})^n u[n] \qquad \cdots 4$$

- ROC 3: $|z| < \frac{1}{2}$ $\cdots 3$

$$x[n] = \frac{4}{3}(-2^n u[-n-1]) - \frac{1}{3}(-(\tfrac{1}{2})^n u[-n-1]) \qquad \cdots 4$$



$\cdots$

ROC: 9 points in total

$x[n]$: 12 points in total.

2. (25) Fast Fourier Transform Algorithm

   a. (10) Derive the complete equations that describe a Fast Fourier Transform (FFT) Algorithm to compute a 10-point Discrete Fourier Transform (DFT).

   b. (15) Draw a complete and fully labeled flow diagram for your 10-point FFT algorithm.

(a) 10-pt DFT = 5 x 2-pt DFT

$$X^{(10)}(k) = \sum_{n=0}^{9} x[n]\, e^{-j\frac{2\pi kn}{10}} \quad , \quad k = 0, \ldots, 9.$$

$$= \sum_{\ell=0}^{4} \sum_{m=0}^{1} x[5m+\ell]\, e^{-j\frac{2\pi k(5m+\ell)}{10}}$$

$$= \sum_{m=0}^{1} x[5m]\, e^{-j\frac{2\pi k\cdot 5m}{10}} + e^{-j\frac{2\pi k}{10}} \sum_{m=0}^{1} x[5m+1]\, e^{-j\frac{2\pi k\cdot 5m}{10}} + e^{-j\frac{4\pi k\cdot 5m}{10}} \sum_{m=0}^{1} x[5m+2] e^{-j\frac{2\pi k\cdot 5m}{10}}$$

$$+ e^{-j\frac{6\pi k}{10}} \sum_{m=0}^{1} x[5m+3] e^{-j\frac{2\pi k\cdot 5m}{10}} + e^{-j\frac{8\pi k}{10}} \sum_{m=0}^{1} x[5m+4]\, e^{-j\frac{2\pi k\cdot 5m}{10}}$$

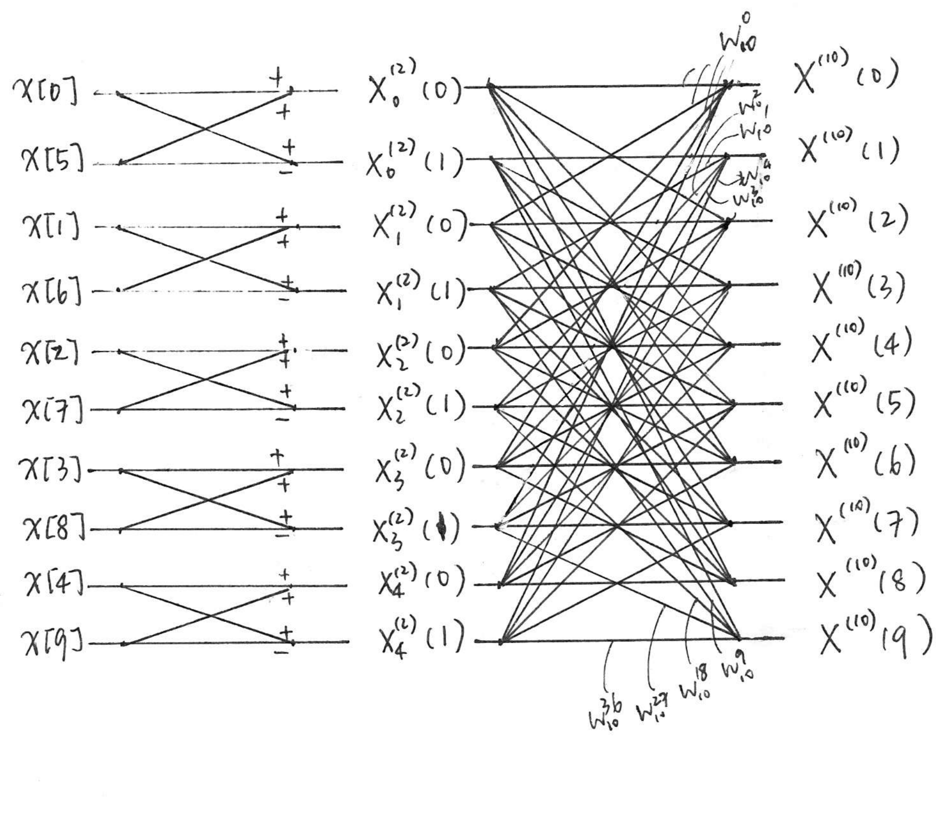
$$= X_0^{(2)}(k') + X_1^{(2)}(k') e^{-j\frac{2\pi k}{10}} + X_2^{(2)}(k') e^{-j\frac{4\pi k}{10}} + X_3^{(2)}(k') e^{-j\frac{6\pi k}{10}} + X_4^{(2)}(k') e^{-j\frac{8\pi k}{10}}$$

$$= X_0^{(2)}(k') + X_1^{(2)}(k')\, W_{10}^{k} + X_2^{(2)}(k')\, W_{10}^{2k} + X_3^{(2)}(k')\, W_{10}^{3k} + X_4^{(2)}(k')\, W_{10}^{4k}$$

$k' = k \bmod 2$

where $W_N^k = e^{-j2\pi k/N}$

(b).



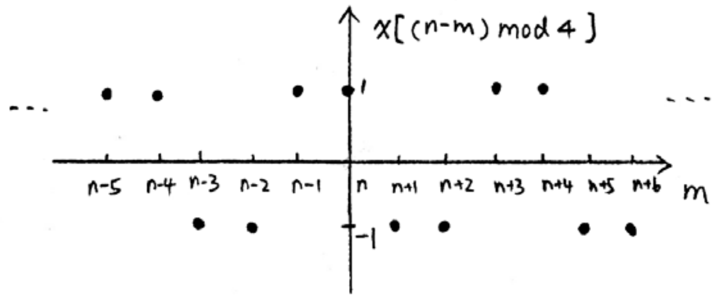
3. (25 pts) Periodic Convolution

a) (21) Determine the 4-point periodic (circular) convolution $w[n]$ of the two 4-point signals $x[n]$ and $y[n]$ shown below:

n	0	1	2	3
$x[n]$	1	1	−1	−1
$y[n]$	1	2	3	4

b) (4) To what length would these signals need to be zero-padded in order that a portion of their periodic convolution will match the non-zero region of their aperiodic convolution?

a) $z[n] = x[n] \circledast y[n] = \sum_{m=0}^{3} x[(n-m) \bmod 4]\, y[m]$



$y[m]$	1	2	3	4	
$x[0-m]$	1	−1	−1	1	$z[0] = 1-2-3+4 = 0$
$x[1-m]$	1	1	−1	−1	$z[1] = 1+2-3-4 = -4$
$x[2-m]$	−1	1	1	−1	$z[2] = -1+2+3-4 = 0$
$x[3-m]$	−1	−1	1	1	$z[3] = -1-2+3+4 = 4$
$x[4-m]$	1	−1	−1	1	$z[4] = 1-2-3+4 = 0$

$\vdots$

$z[n] = z[n \bmod 4]$

(b) In general, if $x[n]$ has length N and $y[n]$ has length M (not include zeros at the start and end of the sequence). Periodic convolution with length $N+M-1$ will match aperiodic convolution results. Since $N=4$, $M=4$ in this problem, we have $N+M-1=7$. Therefore, we need to pad each given sequence with 3 more zeros.

4. (25 pts) Spectral analysis with the DFT. Consider the 10-point signal $x[n] = \cos(3\pi n/10),\ n = 0,\ldots,9$.

c) (10) Find a simple closed-form expression for the 10-point DFT $X^{(10)}[k]$ of $x[n]$ in terms of the function $\mathrm{psinc}_N(\omega) = \dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$.

d) (15) Carefully draw a complete sketch of $X^{(10)}[k],\ k = 0,\ldots,9$ being sure to dimension all important quantities and fully label the axes. You may ignore the contribution of any linear phase terms to your sketch.

a. Let $y[n] = \cos\left(\frac{3\pi n}{10}\right)$, $w[n] = u[n] - u[n-10]$

$$x[n] = y[n] \cdot w[n]$$

$$Y(w) = \mathrm{rep}_{2\pi}\ \pi\left[\delta(w - \tfrac{3}{10}\pi) + \delta(w + \tfrac{3}{10}\pi)\right]$$

$$W(w) = \sum_{n=0}^{9} e^{-jw} = \frac{1 - e^{-j10w}}{1 - e^{-jw}} = \frac{e^{-jw5}(e^{j5w} - e^{-j5w})}{e^{-j\frac{w}{2}}(e^{j\frac{w}{2}} - e^{-j\frac{w}{2}})}$$

$$= e^{-j\frac{9}{2}w}\frac{\sin(5w)}{\sin(\frac{w}{2})} = e^{-j\frac{9}{2}w}\,\mathrm{psinc}_{10}(w)$$

$$X(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\theta) W(w-\theta)\,d\theta$$

$$= \frac{1}{2} e^{-j\frac{9}{2}(w - \frac{3}{10}\pi)}\,\mathrm{psinc}_{10}(w - \tfrac{3}{10}\pi)$$
$$+ \frac{1}{2} e^{-j\frac{9}{2}(w + \frac{3}{10}\pi)}\,\mathrm{psinc}_{10}(w + \tfrac{3}{10}\pi)$$

$$X^{(10)}[k] = X(w)\Big|_{w = \frac{2\pi k}{10}}$$

$$= \frac{1}{2} e^{-j\frac{9}{2}\left(\frac{2\pi k - 3\pi}{10}\right)}\,\mathrm{psinc}_{10}\left(\frac{2\pi k - 3\pi}{10}\right)$$
$$+ \frac{1}{2} e^{-j\frac{9}{2}\left(\frac{2\pi k + 3\pi}{10}\right)}\,\mathrm{psinc}_{10}\left(\frac{2\pi k + 3\pi}{10}\right)$$

5. (continued)