19 February 2014 **Name:**____________________

ECE 438 **Exam No. 1** **Spring 2014**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.

1. (25 pts.) Consider the linear, time-invariant system defined by the difference equation

$$y[n] = x[n] - x[n-1] - y[n-1].$$

   a. (10) Find a simple expression for the frequency response $H(\omega)$ of this system.
   b. (10) Based on your answer to part (a), find simple expressions for the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.
   c. (5) Based on your answer to part (b), carefully sketch the magnitude $|H(\omega)|$ and phase $\underline{/H(\omega)}$ of the frequency response of this system.

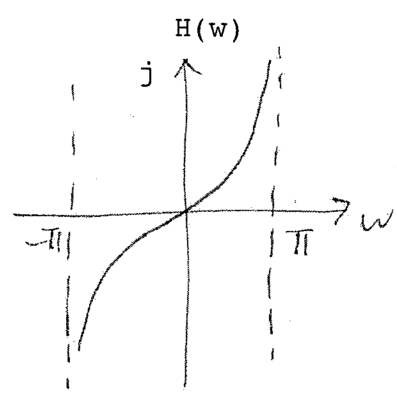
a. Let $x[n] = e^{j\omega n}$, then $y[n] = H(\omega) e^{j\omega n}$

$$H(\omega) e^{j\omega n} = e^{j\omega n} - e^{j\omega(n-1)} - H(\omega) e^{j\omega(n-1)}$$

$$\Rightarrow H(\omega) = 1 - e^{-j\omega} - H(\omega) e^{-j\omega}$$

$$H(\omega) = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)}$$

$$= \frac{2j \sin\frac{\omega}{2}}{2\cos\frac{\omega}{2}} = j \tan\frac{\omega}{2}$$



b. $|H(\omega)| = \left|j \tan\frac{\omega}{2}\right| = \left|\tan\frac{\omega}{2}\right|$

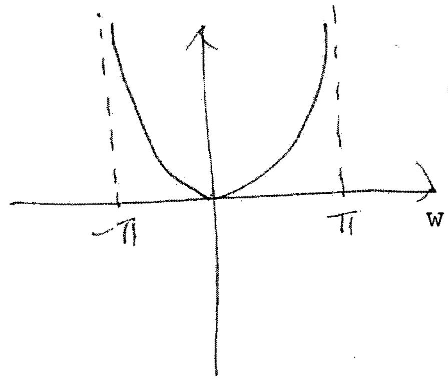
$$\angle H(\omega) = \angle j + \angle \tan\frac{\omega}{2}$$

$$= \begin{cases} \frac{\pi}{2}, & \tan\frac{\omega}{2} > 0 \\ \frac{\pi}{2} - \pi, & \tan\frac{\omega}{2} < 0 \end{cases}$$

$$= \begin{cases} \frac{\pi}{2} & 0 < \omega < \pi \\ -\frac{\pi}{2} & -\pi < \omega < 0 \end{cases} \quad 2\pi \text{ periodic}$$

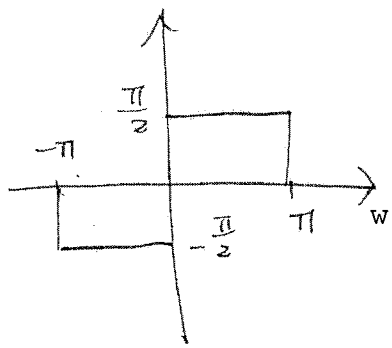
1. (continued)

c. $|H(w)|$:



2π periodic

$\angle H(w)$



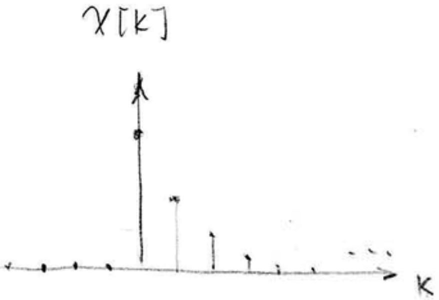
2π periodic

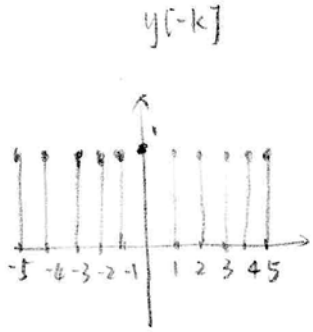
2. (25 pts.) Perform the convolution $w[n]$ of the following two signals, and carefully sketch the output signal

$$x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & \text{else} \end{cases}, \text{ and } \quad y[n] = \begin{cases} 1, & -5 \leq n \leq 5 \\ 0, & \text{else} \end{cases}.$$

2. $w[n] = x[n] * y[n]$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot y[n-k]$$





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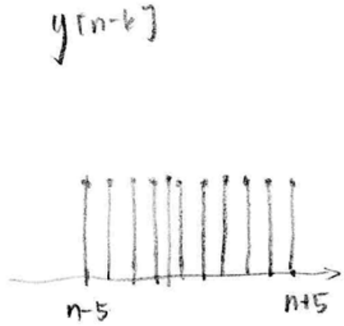


case 1: $n+5 < 0$ or $n < -5$, $w[n] = 0$

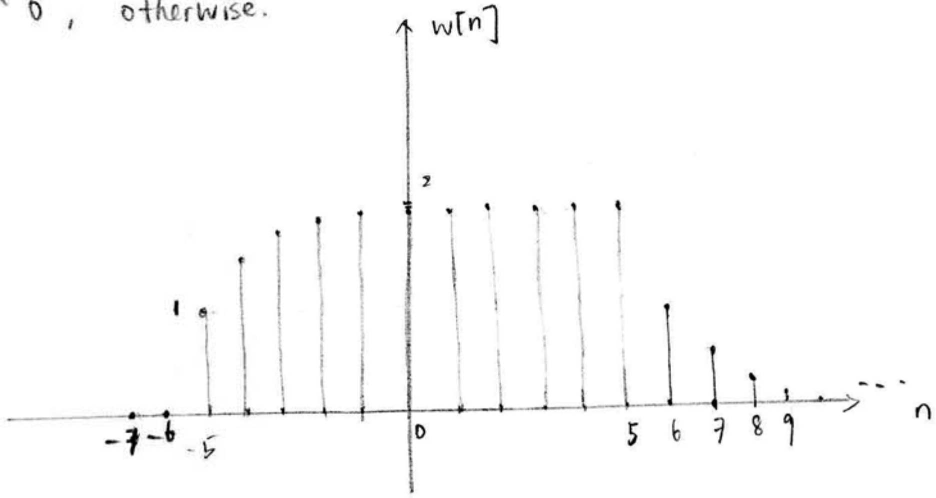
case 2: $n+5 \geq 0$, $n-5 \leq 0$ or $-5 \leq n \leq 5$

$$w[n] = \sum_{k=0}^{n+5} \left(\tfrac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{n+6}}{1-\frac{1}{2}} = 2 - \left(\tfrac{1}{2}\right)^{n+5}$$

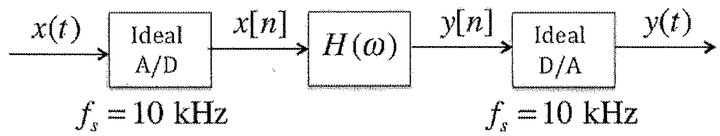
case 3: $n-5 > 0$ or $n > 5$

$$w[n] = \sum_{k=n-5}^{n+5} \left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right)^{n-5} \frac{\left(1-\left(\frac{1}{2}\right)^{11}\right)}{\left(1-\frac{1}{2}\right)} = \left(\tfrac{1}{2}\right)^{n-6} - \left(\tfrac{1}{2}\right)^{n+5}$$

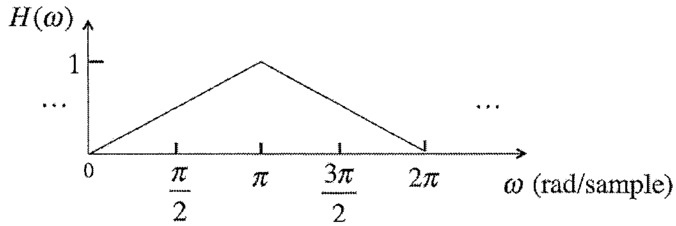
$$\Rightarrow \quad w[n] = \begin{cases} 2-\left(\frac{1}{2}\right)^{n+5}, & -5 \leq n \leq 5 \\ \left(\frac{1}{2}\right)^{n-6} - \left(\frac{1}{2}\right)^{n+5}, & n > 5 \\ 0, & \text{otherwise.} \end{cases}$$



3. (25) Consider the system shown below, which operates at a 10 kHz sampling rate.



Here the digital filter in the center of the system has frequency response $H(\omega)$ given below.



Suppose that the input to this system is

$$x(t) = 1 + \cos\left(2\pi(1000)t\right) + \cos\left(2\pi(4000)t\right).$$

Note that the Ideal A/D can be viewed as an ideal low pass filter with cutoff $f_c = 5$ kHz, followed by a comb operation sampling at a 10^{-4} sec. interval. Similarly, the Ideal D/A can be viewed as a train of sample-modulated impulses separated by a 10^{-4} sec. interval (comb operation), followed by an ideal low pass filter with cutoff $f_c = 5$ kHz.

a. (8) Find a simple expression for the DTFT $X(\omega)$ of the signal $x[n]$.

b. (9) Find a simple expression for the DTFT $Y(\omega)$ of the signal $y[n]$.

c. (8) Find a simple expression for the output $y(t)$ of the overall system.

3. **(continued - 1)**

$$x(t) = 1 + \cos(2\pi(1000)t) + \cos(2\pi(4000)t)$$

$$x(t) \xrightarrow{CTFT} X(f)$$

then $X(f) = \delta(f) + \frac{1}{2}[\delta(f-1000) + \delta(f+1000)] + \frac{1}{2}[\delta(f-4000) + \delta(f+4000)]$

(a) $x[n] = x\left(\frac{n}{f_s}\right) \qquad X(\omega) = f_s \, rep_{f_s}\left[X\left(\frac{\omega}{2\pi} f_s\right)\right]$

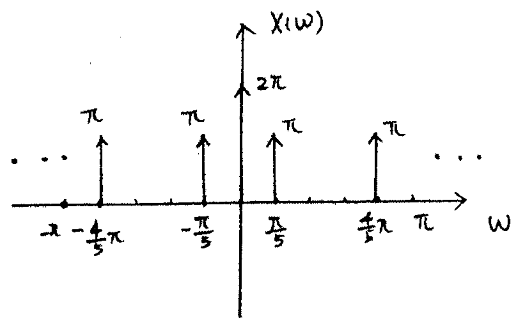
$$\Rightarrow X(\omega) = f_s \, rep_{f_s}\left[\delta\left(\frac{\omega}{2\pi}f_s\right) + \frac{1}{2}\delta\left(\frac{\omega}{2\pi}f_s - 1000\right) + \frac{1}{2}\delta\left(\frac{\omega}{2\pi}f_s + 1000\right) + \frac{1}{2}\delta\left(\frac{\omega}{2\pi}f_s - 4000\right) + \frac{1}{2}\delta\left(\frac{\omega}{2\pi}f_s + 4000\right)\right]$$

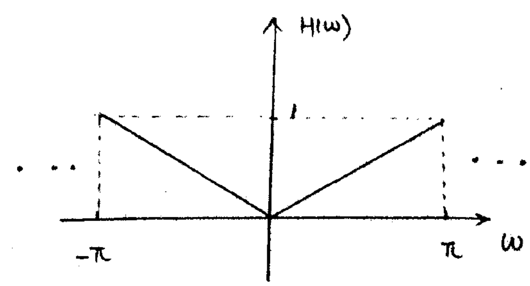
since $\delta\left(\frac{\omega}{2\pi}f_s\right) = \frac{2\pi}{f_s}\delta(\omega)$

$$X(\omega) = rep_{2\pi}\left[2\pi\delta(\omega) + \pi\delta\left(\omega - \frac{2000\pi}{f_s}\right) + \pi\delta\left(\omega + \frac{2000\pi}{f_s}\right) + \pi\delta\left(\omega - \frac{8000\pi}{f_s}\right) + \pi\delta\left(\omega + \frac{8000\pi}{f_s}\right)\right]$$

$$\boxed{X(\omega) = rep_{2\pi}\left[2\pi\delta(\omega) + \pi\delta\left(\omega - \frac{\pi}{5}\right) + \pi\delta\left(\omega + \frac{\pi}{5}\right) + \pi\delta\left(\omega - \frac{4}{5}\pi\right) + \pi\delta\left(\omega + \frac{4}{5}\pi\right)\right]}$$

(b)

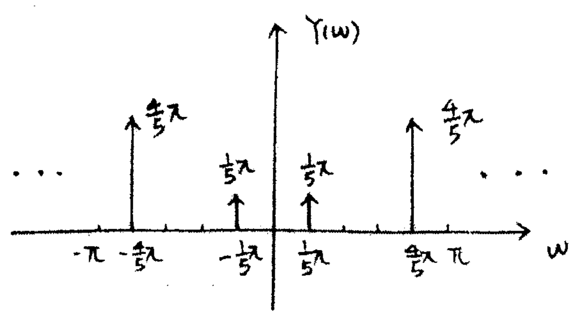




$H(\omega)\big|_{\omega=\frac{\pi}{5}} = \frac{1}{5} \qquad H(\omega)\big|_{\omega=\frac{4}{5}\pi} = \frac{4}{5}$

$H(\omega)\big|_{\omega=-\frac{\pi}{5}} = \frac{1}{5} \qquad H(\omega)\big|_{\omega=-\frac{4}{5}\pi} = \frac{4}{5}$

$Y(\omega) = X(\omega)H(\omega)$



$$\boxed{Y(\omega) = rep_{2\pi}\left[\frac{1}{5}\pi\delta\left(\omega - \frac{1}{5}\pi\right) + \frac{1}{5}\pi\delta\left(\omega + \frac{1}{5}\pi\right) + \frac{4}{5}\pi\delta\left(\omega - \frac{4}{5}\pi\right) + \frac{4}{5}\pi\delta\left(\omega + \frac{4}{5}\pi\right)\right]}$$

3. (continued - 2)

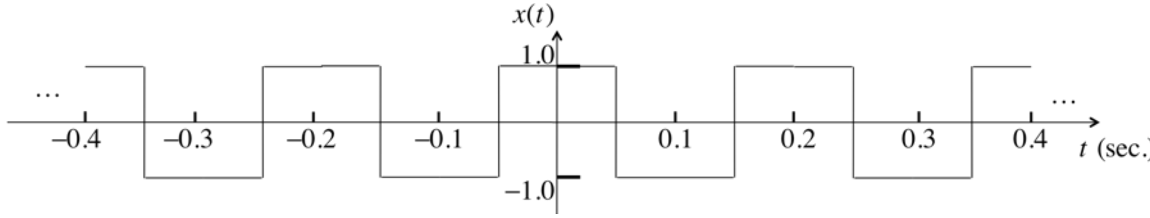
(c) $Y(f) = \mathcal{F}\{y(t)\} = Y(\omega)\Big|_{\omega = \frac{2\pi f}{f_s}}$ and filtered with LPF of cut-off frequency 5KHZ

$$\Rightarrow Y(f) = \frac{1}{10}\delta\left(f - \frac{f_s}{10}\right) + \frac{1}{10}\delta\left(f + \frac{f_s}{10}\right) + \frac{2}{5}\delta\left(f - \frac{2}{5}f_s\right) + \frac{2}{5}\delta\left(f + \frac{2}{5}f_s\right)$$

$$= \frac{1}{10}\delta(f - 1000) + \frac{1}{10}\delta(f + 1000) + \frac{2}{5}\delta(f - 4000) + \frac{2}{5}\delta(f + 4000)$$

$$\boxed{y(t) = \frac{1}{5}\cos(2\pi(1000)t) + \frac{4}{5}\cos(2\pi(4000)t)}$$

4. (25 pts) Consider the signal $x(t)$ shown below:

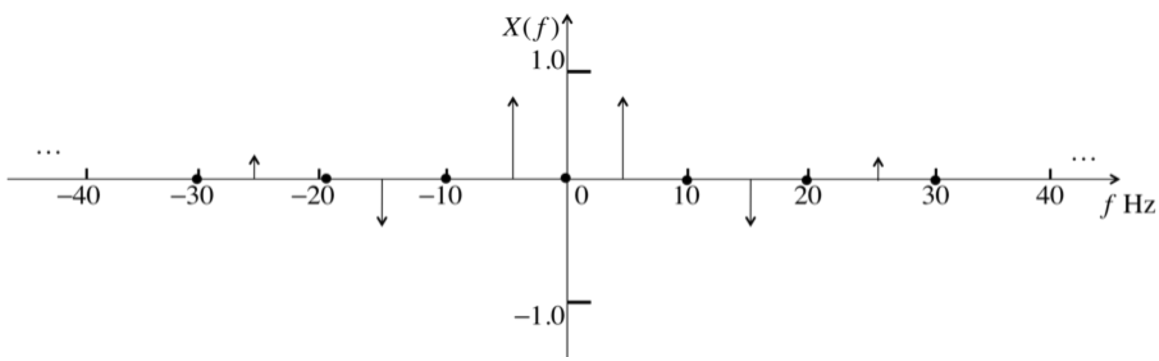


a) (11) Find a simple closed-form expression for the CTFT $X(f)$ of $x(t)$. Your answer should not contain any operators, i.e. your answer should be written in summation form without comb or rep operators.

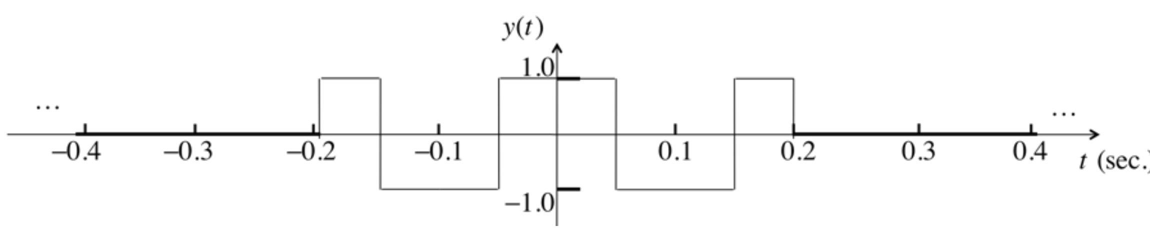
$$x(t) = 2\cdot \mathrm{rep}_{0.2}\left[\mathrm{rect}(t/0.1)\right] - 1$$

$$\begin{aligned} X(f) &= 2\frac{1}{0.2}\cdot \mathrm{comb}_{\frac{1}{0.2}}\left[0.1\cdot \mathrm{sinc}(0.1f)\right] - \delta(f) \\ &= \mathrm{comb}_{0.2}\left[\mathrm{sinc}(0.1f)\right] - \delta(f) \\ &= \sum_{k=-\infty}^{\infty} \mathrm{sinc}(0.1k/0.2)\delta(f - k/0.2) - \delta(f) \\ &= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \mathrm{sinc}(k/2)\delta(f - k/0.2) \end{aligned}$$

b) (5) Carefully sketch $X(f)$. Be sure to dimension all important quantities.



Consider the signal shown below:



c) (6) Find a simple closed-form expression for the CTFT $Y(f)$ of $y(t)$. Your answer should not contain any operators.

$$y(t) = x(t) \cdot \text{rect}(t/0.4)$$

$$Y(f) = X(f) * 0.4 \cdot \text{sinc}(0.4f)$$

$$= 0.4 \cdot \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \text{sinc}(k/2) \cdot \text{sinc}\big(0.4(f - k/0.2)\big)$$

d) (3) Carefully sketch $Y(f)$. Be sure to dimension all important quantities.