1 May 2013 **Name:_______________ Signature: _________________**

ECE 438 **Final Exam** **Spring 2013**

- You have 120 minutes to work the following five problems.
- Be sure to show **all** your work to obtain full credit.
- You do *not* need to derive any result that can be found on the formula sheet. However, you should state that it can be found there.
- The exam is closed book and closed notes.
- Calculators are **not** permitted.
- It will be to your advantage to budget your time so that you can write something for each problem. Please note that the problems are arranged in the order that the topics were covered during the semester, not necessarily in the order of difficulty to solve them.

1. (25 pts.) Consider a system described by the following equation

$$y[n] = \frac{1}{4}\left(x[n] + 2x[n-1] + x[n-2]\right)$$

.

a. (8) Find the response $y[n]$ to the following input

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{else} \end{cases}$$

b. (9) Find a simple expression for the frequency response $H(\omega)$ for this system.

c. (8) From your answer to part (b), determine simple expressions for the magnitude and phase of the frequency response.

Solution to Problem 1.

a)

n	0	1	2	3	4	5	6	7
$x[n]$	1	1	1	1	1	0	0	0
$2x[n-1]$	0	2	2	2	2	2	0	0
$x[n-2]$	0	0	1	1	1	1	1	0

$$\therefore y[n] = \frac{1}{4}\left(x[n] + 2x[n-1] + x[n-2]\right)$$

$$= \{\underset{\underset{n=0}{\uparrow}}{\tfrac{1}{4}}, \tfrac{3}{4}, 1, 1, 1, \tfrac{3}{4}, \tfrac{1}{4}\}$$

It might be good to state explicitly that y[n] = 0 outside this range.

b). $h[n] = \frac{1}{4}\left(\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$

$H(w) = \frac{1}{4}\left(1 + 2e^{-jw} + e^{-2jw}\right)$

$$\Rightarrow \boxed{H(w) = \frac{1}{4}\left(1+e^{-jw}\right)^2}$$

c) $|H(w)|^2 = H(w)H^*(w) = \frac{1}{16}(1+e^{-jw})^2(1+e^{jw})^2$

$= \frac{1}{16}(2 + e^{-jw} + e^{jw})^2$

$= \frac{1}{16}(2 + 2\cos w)^2$

$= \frac{1}{4}(1+\cos w)^2$

$\therefore |H(w)| = \frac{1}{2}(1+\cos w)$

$\Rightarrow \boxed{|H(w)| = \cos^2(w/2)}$

Alternate approach based on factoring out the half-angle.

$H(w) = \left(\frac{1+e^{-jw}}{2}\right)^2 = \left[e^{-j\frac{w}{2}}\left(\frac{e^{j\frac{w}{2}} + e^{-j\frac{w}{2}}}{2}\right)\right]^2$

$= e^{-jw}\cos^2\left(\frac{w}{2}\right)$

$$\therefore \boxed{\angle H(w) = -w, \quad -\pi < w \leq \pi}$$

+ angle(cos^2(omega/2)) = - omega + 0

2. (25 pts.) Let $x[n], n = 0,...,N-1$ denote an N-point signal with N-point DFT

$$X^{(N)}[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}.$$

Let $y[n], n = 0,...,2N-1$ denote a new $2N$-point signal defined according to

$$y[n] = \begin{cases} x[n], & n = 0,...,N-1, \\ x[n-N], & n = N,...,2N-1. \end{cases}$$

Let $Y^{(2N)}[k]$ denote the $2N$-point DFT of $y[n]$.

a. (15) Find a simple expression for $Y^{(2N)}[k], k = 0,...,2N-1$ in terms of $X^{(N)}[l], l = 0,...,N-1$.

Now let $x[n], n = 0,...,N-1$ be defined as before. But let $y[n], n = 0,...,2N-1$ denote a new $2N$-point signal defined according to

n	0	1	2	3	4	5	...	$2N-2$	$2N-1$
$y[n]$	$x[0]$	$x[0]$	$x[1]$	$x[1]$	$x[2]$	$x[2]$	...	$x[N-1]$	$x[N-1]$

b. (10) Find a simple expression for $Y^{(2N)}[k], k = 0,...,2N-1$ in terms of $X^{(N)}[l], l = 0,...,N-1$.

2. (continued – 1)

a) $\because$ $y[n] = \begin{cases} x[n], & n = 0, \dots, N-1, \\ x[n-N], & n = N, \dots, 2N-1 \end{cases}$

$\therefore$ $Y^{(2N)}[k]$

$$= \sum_{n=0}^{2N-1} y[n] e^{-j2\pi kn/(2N)}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/(2N)} + \sum_{n=N}^{2N-1} x[n-N] e^{-j2\pi kn/(2N)}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{\frac{k}{2}n}{N}} + \sum_{m=0}^{N-1} x[m] e^{-j2\pi k(m+N)/(2N)}$$

$$= X^{(N)}[k/2] + \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi km}{2N}} \cdot e^{-j\frac{2\pi kN}{2N}}$$

$$= X^{(N)}[k/2] + X^{(N)}[k/2] \cdot e^{-jk\pi}$$

$$= (1 + e^{-jk\pi}) \underbrace{X^{(N)}[k/2]}$$

$$\underbrace{\sum_{m=0}^{N-1} x[m] e^{-j2\pi(k/2)/N}}$$

$$= \begin{cases} 2X^{(N)}[k/2], & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases}$$

Strictly speaking, the expression pointed to by the left arrow is not correct, because X^(N)[k/2] is only defined for even k. So until the separate cases are evaluated as they are after the equal sign on the following line, the expression X^(N)[k/2] should be replaced by the expression pointed to by the right arrow.

b) $y[n] = x[\lfloor n/2 \rfloor]$, $n = 0, 1, \cdots, 2N-1$

Note: $\lfloor x \rfloor$ is floor function.

$\lfloor x \rfloor = \max\{ m \in \mathbb{Z} \mid m \le x \}$.

$\therefore\ Y^{(2N)}[k]$

$$= \sum_{n=0}^{2N-1} y[n]\, e^{-j2\pi kn/(2N)}$$

$$= \sum_{n=0}^{2N-1} x[\lfloor n/2 \rfloor]\, e^{-j2\pi kn/(2N)}$$

$$= \underbrace{\sum_{n=0}^{2N-2} x[\tfrac{n}{2}]\, e^{-j2\pi kn/(2N)}}_{n \text{ is even}} + \underbrace{\sum_{n=1}^{2N-1} x[\tfrac{n-1}{2}]\, e^{-j2\pi kn/(2N)}}_{n \text{ is odd.}}$$

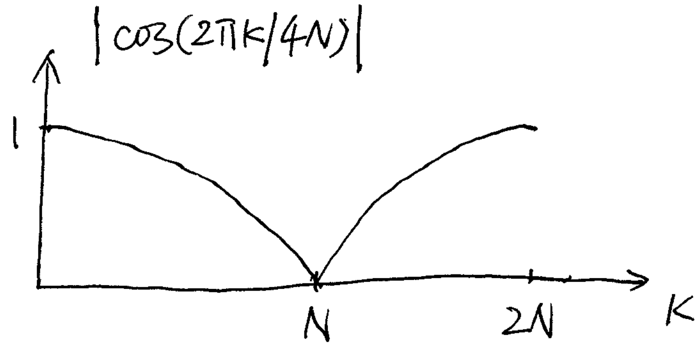
$$= \sum_{m=0}^{N-1} x[m]\, e^{-j2\pi k\cdot 2m/(2N)} + \sum_{l=0}^{N-1} x[l]\, e^{-j2\pi k(2l+1)/(2N)}$$

$$= X^{(N)}[k] + \sum_{l=0}^{N-1} x[l]\, e^{-j2\pi k\cdot 2l/(2N)} \cdot e^{-j2\pi k/(2N)}$$

$$= X^{(N)}[k] + X^{(N)}[k] \cdot e^{-jk\pi/N}$$

$$= (1 + e^{-jk\pi/N})\, X^{(N)}[k]$$

$$= 2e^{-j\pi k/(2N)} \cos(2\pi k/4N)\, X^{(N)}[k]$$



3. (25) Consider a random variable X with probability density function

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

Suppose we quantize this random variable to generate a new random variable $Y = Q(X)$, according to the quantizer

$$Q(x) = \begin{cases} \frac{1}{4}, & 0 \le x < \frac{1}{2}, \\ \frac{3}{4}, & \frac{1}{2} \le x \le 1. \end{cases}$$

a. (6) Find the mean $E(Y)$ and variance σ_Y^2.

b. (9) Find the correlation coefficient ρ_{XY} between X and Y.

c. (10) Find the exact mean-squared quantization error.

$$\varepsilon = E\left((Y - X)^2\right).$$

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$$E[Y]=E[Q(X)]=\int_{-\infty}^{\infty}Q(x)f_X(x)\,dx$$

$$=\int_0^{\frac{1}{2}}\frac{1}{4}\cdot 2x\,dx+\int_{\frac{1}{2}}^{1}\frac{3}{4}\cdot 2x\,dx$$

$$=\frac{1}{4}x^2\Big|_0^{\frac{1}{2}}+\frac{3}{4}x^2\Big|_{\frac{1}{2}}^{1}=\frac{5}{8}$$

$$\sigma_Y^2=E[(Y-E[Y])^2]=E[(Q(x)-\tfrac{5}{8})^2]=\int_{-\infty}^{\infty}(Q(x)-\tfrac{5}{8})^2f_X(x)\,dx$$

$$=\int_0^{\frac{1}{2}}(\tfrac{1}{4}-\tfrac{5}{8})^2\cdot 2x\,dx+\int_{\frac{1}{2}}^{1}(\tfrac{3}{4}-\tfrac{5}{8})^2\cdot 2x\,dx$$

$$=\frac{9}{64}x^2\Big|_0^{\frac{1}{2}}+\frac{1}{64}x^2\Big|_{\frac{1}{2}}^{1}=\frac{3}{64}$$

An alternative approach would be to use sigma^2_Y = E (Y^2) - (E(Y))^2.

b. $$E[XY]=E[XQ(X)]=\int_{-\infty}^{\infty}xQ(x)f_X(x)\,dx$$

$$=\int_0^{\frac{1}{2}}\frac{1}{4}x\cdot 2x\,dx+\int_{\frac{1}{2}}^{1}\frac{3}{4}x\cdot 2x\,dx$$

$$=\frac{1}{6}x^3\Big|_0^{\frac{1}{2}}+\frac{1}{2}x^3\Big|_{\frac{1}{2}}^{1}=\frac{11}{24}$$

$$E[X]=\int_{-\infty}^{\infty}xf_X(x)\,dx=\int_0^1 2x^2\,dx=\frac{2}{3}x^3\Big|_0^1=\frac{2}{3}$$

$$\sigma_X^2=\int_{-\infty}^{\infty}(x-E[X])^2f_X(x)\,dx=\int_0^1(x-\tfrac{2}{3})^2\cdot 2x\,dx$$

$$=\int_0^1(2x^3-\tfrac{8}{3}x^2+\tfrac{8}{9}x)\,dx=\frac{1}{2}x^4-\frac{8}{9}x^3+\frac{4}{9}x^2\Big|_0^1=\frac{1}{18}$$

3. (continued – 1)

$$\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{\frac{11}{24} - \frac{2}{3}\times\frac{5}{8}}{\sqrt{\frac{1}{18}\times\frac{3}{64}}} = \frac{\sqrt{6}}{3}$$

c. $\varepsilon = E[(Y-X)^2] = E[(Q(X)-X)^2]$

$$= \int_{-\infty}^{\infty} (Q(x)-x)^2 f_X(x)\,dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{1}{4}-x\right)^2 \cdot 2x\,dx + \int_{\frac{1}{2}}^{1} \left(\frac{3}{4}-x\right)^2 \cdot 2x\,dx$$

$$= \int_0^{\frac{1}{2}} \left(2x^3 - x^2 + \frac{1}{8}x\right) dx + \int_{\frac{1}{2}}^{1} \left(2x^3 - 3x^2 + \frac{9}{8}x\right) dx$$

$$= \frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{16}x^2 \Big|_0^{\frac{1}{2}} + \frac{1}{2}x^4 - x^3 + \frac{9}{16}x^2 \Big|_{\frac{1}{2}}^{1}$$

$$= \frac{1}{48}$$

4. (25 pts) Consider the 2-D function $f(x,y) = \cos\big(2\pi(x+y)\big)\text{rect}\left(\frac{x}{10},\frac{y}{10}\right)$.

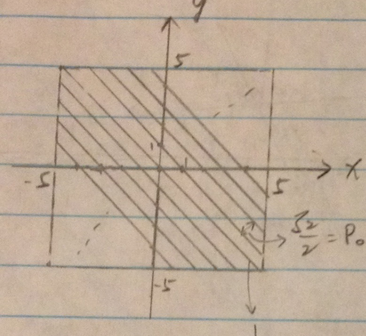
   a. (7) Sketch $f(x,y)$ with enough detail to show that you know what it looks like. Be sure to dimension your axes.

   b. (10) Find a simple expression for the continuous-space Fourier transform (CSFT) $F(u,v)$ of $f(x,y)$. Your answer should not contain any operators other than summations.

   c. (8) Sketch $F(u,v)$ with enough detail to show that you know what it looks like. Be sure to dimension your axes.

3.(a)

$$f(x,y) = \cos(2\pi(x+y))\,\mathrm{rect}\left(\frac{x}{10}, \frac{y}{10}\right)$$



$$\frac{1}{P_0} = \sqrt{1^2+1^2}$$

$$P_0 = \frac{\sqrt{2}}{2}$$

(b)

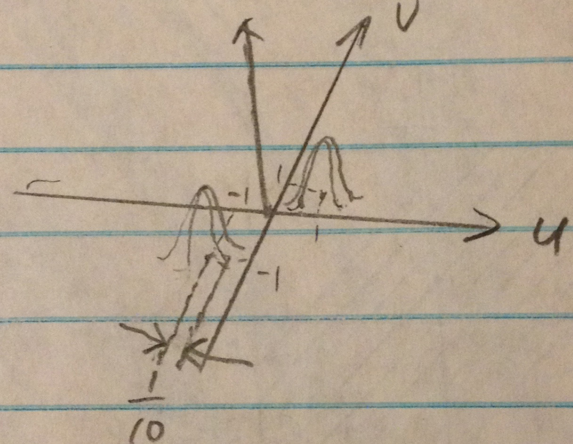
$$f(x,y) = \cos(2\pi(x+y))\,\mathrm{rect}\left(\frac{x}{10}, \frac{y}{10}\right)$$

$$\updownarrow \text{ CSFT}$$

$$F(u,v) = \frac{1}{2}\left(\delta(u-1, v-1) + \delta(u+1, v+1)\right) * 10^2\,\mathrm{sinc}(10u, 10v)$$

$$= 50\left[\mathrm{sinc}(10(u-1), 10(v-1)) + \mathrm{sinc}(10(u+1), 10(v+1))\right]$$

3. (c) Sketch $F(u,v)$



5. (25 pts) Consider a spatial filter with point spread function $h[m,n]$ given below

	$h[m,n]$	$n=-1$	$n=0$	$n=1$
m	1	−1	0	1
	0	0	1	0
	−1	1	0	−1

a. (10) Find the output $g[m,n]$ when this filter is applied to the following input image. You may assume that the boundary pixel values are extended beyond the boundary. You need only calculate the output over the original 11×11 set of pixels in the input image.

0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	1	0
0	1	1	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

b. (12) Find a simple expression for the frequency response $H(\mu,\nu)$ of this filter, and sketch the magnitude $|H(\mu,\nu)|$ along the μ axis, the ν axis, the $\mu=\nu$ axis, and the $\mu=-\nu$ axis.

c. (3) Using your results from parts a) and b), explain what this filter does. Relate spatial domain properties to frequency domain properties. Be sure to examine what happens at each edge of the region of 1's above, and how this relates to the frequency domain, as well as what happens in the center of the region of 1's and in the border of 0's that surrounds the non-zero portion of the image.

5(a) The output $g(m,n)$:

$$
\begin{matrix}
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 \\
0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & 0 \\
0 & 1 & 1 & 1 & 1 & 2 & 2 & -1 & -1 & 0 & 0 \\
1 & 2 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0
\end{matrix}
$$

(b) $$H(\mu,\nu) = \sum_{m=-1}^{1}\sum_{n=-1}^{1} h[m,n]\, e^{-j(\mu m+\nu n)}$$

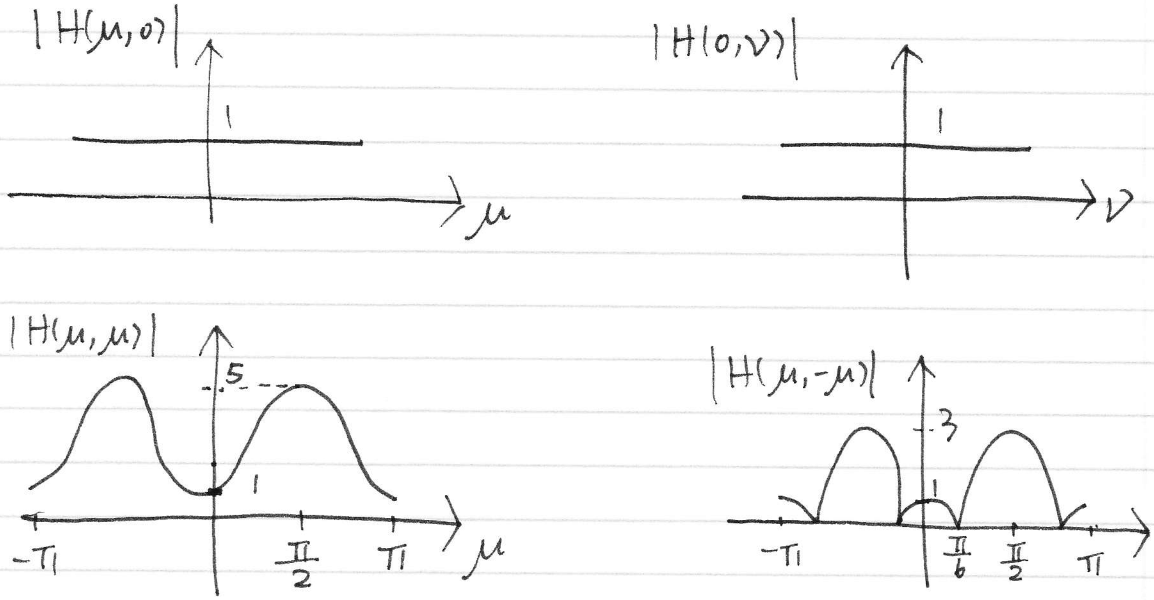
$$= -e^{j(\mu+\nu)} + e^{-j\mu+j\nu} + e^{j\mu-j\nu} - e^{-j\mu-j\nu} + 1$$

Along the μ axis : $\nu=0$, $H(\mu,0) = -e^{j\mu} + e^{-j\mu} + e^{j\mu} - e^{-j\mu} + 1$
$= 1$

Along the ν axis : $\mu=0$, $H(0,\nu) = -e^{j\nu} + e^{j\nu} + e^{-j\nu} - e^{-j\nu} + 1$
$= 1$

Along the $\mu=\nu$ axis : $H(\mu,\mu) = -e^{j2\mu} - e^{-j2\mu} + 1 + 1 + 1$
$= 3 - 2\cos 2\mu$

Along the $\mu=-\nu$ axis : $H(\mu,-\mu) = -1 + e^{-j2\mu} + e^{j2\mu} - 1 + 1$
$= 2\cos 2\mu - 1$



(c) The filter doesn't affect the frequency along the horizontal and vertical direction.
Along the $m=n$ diagonal, the mid-frequency component is enhanced. The edge is emphasized.
Along the $m=-n$ diagonal, the mid-frequency component is also enhanced. But the component at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are suppressed.
Since the DC response is 1, the center region of 1s is maintained.